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Hamilton-Jacobi Treatment of Fields with Constraints

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إهداء

إنني أهدي رسالتي :

(والداي الغاليان)

إلى الروح التي أحيا بها و الدم الذي يجري في عروقي.
إلى كل عين لم تغمض عن رؤية الجريمة.
إلى كل لسان يخشى يوم لقاء ربه.
إلى كل ابتسامة لم تعرف السخرية.
إلى كل أذن لم تتجاهل سماع الحقيقة.
إلى كل ضمير لم يغفل أبدا.
إلى كل قلب تسكنه الرحمة والطهارة والنية الحسنة.
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إلى كل من يخاف الخالق لا المخلوق.
إلى كل من لا يخاف في الله لومه لائم.
إلى كل من يحكم عقله وضميره وليس قلبه ومشاعره.
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Abstract

Hamilton-Jacobi Treatment of fields with constraints

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In this thesis the basic formalism for treating constrained Hamiltonian systems of field theory are discussed within the framework of two methods, Dirac's and Hamilton-Jacobi method.

Lagrangian for a fermionic and a scalar field, the scalar field coupled to two flavours of fermions through Yukawa couplings and non-Abelian theory of fermions interacting with gauge bosons as an application of non-Abelian Yang-Mills theories are treated as constrained systems using the Hamilton-Jacobi approach. The equations of motion are obtained as total differential equations in many variables. These equations of motion are in exact agreement with those equations that had been obtained using Dirac's method.

Path integral quantization of the coupled scalar field minimally to the vector potential, is discussed as an application of field theory containing first-class constraints only, and the quantization of the relativistic local free field with linear velocity of dimension D containing both first and second-class constraints. Also Hamilton-Jacobi quantization of electromagnetic field coupled to a spinor is studied.

ملخص

معالجة هاملتون جاكوبي لأنظمة المجال المقيدة

إعداد

الطالبة / ولاء إبراهيم إشرم

إشراف

الدكتور/ ناصر فرحات

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إن هذه الأطروحة تعالج بعض أنظمة المجال المقيدة باستخدام طريقتين، طريقة ديراك وطريقة هاملتون جاكوبي (القانونية). الطريقتان تمثلان المعالجة الهاملتونية المقيدة. ففي طريقة ديراك تعرف القيود الأولية ثم يتم تكوين الهاملتون الكلي وتفحص شروط التكامل. أما طريقة الهاملتون جاكوبي (القانونية) فمعادلات الحركة الموصوفة هي معادلات تفاضلية كاملة في عدة متغيرات، وتكون هذه المعادلات قابلة للتكامل إذا تحققت شروط التكامل وبحل هذه المعادلات نحصل على المجالات قيد الدراسة، والتي تصف مسارات النظام. تم معالجة ثلاث نماذج فيزيائية باستخدام الطريقتين المذكورتين أعلاه معالجة كلاسيكية "للاجرانج مجال اللاموجة مع الفيرمونك ومجال اللاموجة مربوط صفتين مميزتين للفيرمونز من خلال ارتباط يوكاوا ونظرية يانج ميلز" وقد وجدت النتائج متطابقة في الحالتين وهذا يشير إلى صحة الدراسة.

التطوير في التكامل الخطي القانوني هو انه لا يفرق بين القيود الأولية والثانوية ولا يحتاج لتعريف معاملات اللاجرانج واستخدام دالة دلتا. كما انه ليس هناك داعي لاستخدام شروط تثبيت المقياس حيث كل ما نريده هو المعادلات التفاضلية الجزئية الجاكوبية ومعادلات الحركة فإذا كان النظام متكاملًا فإنه يمكن تكوين الأبعاد القانونية المختزلة.

لقد تم تكميم نظامين لمجال مقيد بواسطة الطريقتين المذكورتين أعلاه، مجال اللاموجة المترابط بضعف مع جهد كمية موجهة وهذا النظام يحتوي على قيود أولية فقط ومجال حر محلي نسبي مع سرعة خطية ذات بعد D يحتوي على قيود أولية وثانوية ووجدنا أن طريقة التكامل الخطي

لدراك تفرق بين الأنظمة التي تحتوي على قيود أولية فقط وبين أولية وثانوية حيث لكل منها طريقة تكميم مختلفة عن الأخرى.

التكميم بواسطة التكامل الخطي للهاملتون جاكوبي (القانوني) لمجال مغناطيسي مترابط مع المجال المغزلي. وقد تم تحقيق شروط التكامل، وبالتالي النظام متكامل وفي هذه الحالة التكامل الخطي أصبح مباشرة تكامل على محاور الأبعاد المختزلة. وهذا يشير إلى أن هذه الطريقة أسهل من طريقة ديراك.

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Chapter 1

Introduction

1.1 Historical Remarks

All familiar with the standard formulations of quantum mechanics, developed more or less concurrently by Schrödinger, Heisenberg and others in the 1920s, and shown to be equivalent to one another soon thereafter.

In 1933, Dirac made the observation that action plays a central role in classical mechanics (he considered the Lagrangian formulation of classical mechanics to be more fundamental than the Hamiltonian one), but that it seemed to have no important role in quantum mechanics as it was known at that time. He speculated on how this situation might be rectified, and he arrived at conclusion that the propagator in quantum mechanics

"corresponds to" $\exp iS/\hbar$, where S is the classical action evaluated along the classical path.

In 1948, Feynman developed Dirac's suggestion, and succeeded in deriving a third formulation of quantum mechanics, based on the fact that the propagator can be written as a sum over all possible paths (not just the classical one) between the initial and final points. Each path contributes $\exp iS/\hbar$ to the propagator. So while Dirac considered only the classical path, Feynman showed that all paths contribute, in a sense, the quantum particle takes all paths, and the amplitudes for each path add according to the usual quantum mechanical rule for combining amplitudes.

The study of constrained systems was initiated by Dirac [1,2], who set up a formalism for treating singular systems and the constraints involved. He showed that, in the presence of constraints, the number of degrees of freedom of the dynamical system was reduced. His approach was subsequently extended to continuous systems [3].

Following Dirac, There is another approach for quantizing constrained systems of classical singular theories, which was initiated by Feynman kernel [4,5], who first set up a formalism of

the path integral quantization. Faddeev and Popov [6,7] handle constraints in the path integral formalism when only first-class constraints in the canonical gauge are present. The generalization of the method to theories with second-class constraints is given by Senjanovic [8]. Fradkin and Vilkovisky [9,10] re-derived both results in a broader context, where they improved Faddeev's procedure mainly to include covariant constraints; also they extended this procedure to the Gressman variables. When the dynamical system possesses some second-class constraints there exists another method given by Batalin and Fradkin [11]: the BFV-BRST operator quantization method. More, Gitman and Tyutin [12] discussed the canonical quantization of singular theories as well as the Hamiltonian formalism of gauge theories in an arbitrary gauge. An alternative approach was developed by Bukenhout, Sprague and Faddeev [13,14] without following Dirac step by step. In this formalism there is no need to distinguish between first and second-class or primary and secondary constraints, where the primary constraint is a set of relations connection between the momenta and the coordinates. The general formalism is then applied to several problems, quantization of the massive Yang-Mills field theory,

Light-Cone quantization of the self interacting scalar field, and quantization of a local field theory of magnetic monopolies, etc.

A most powerful approach for treating constrained systems is the Hamilton-Jacobi approach [15,16] which has been developed to investigate the constrained systems. Several constrained systems were investigated by using the Hamilton-Jacobi approach [17-22]. The equivalent Lagrangian method is used to obtain the set of Hamilton-Jacobi Partial Differential Equations (HJPDE). In this approach, the distinction between the first- and second-class constraints is not necessary. The equations of motion are written as total differential equations in many variables, which require the investigation of the integrability conditions. In other words, the integrability conditions may lead to new constraints. Moreover, it is shown that gauge fixing, which is an essential procedure to study singular system by Dirac's method, which is not necessary if the Hamilton-Jacobi approach is used.

Following Hamilton-Jacobi approach, there is another approach for quantizing constrained systems of classical singular theories by path integral quantization [23-26].

There are some fields of the constrained system which are

very familiar, like the Electromagnetic theory, Einstein Gravitational theory, Yang-Mills theory, String theory, and Parameterizations-invariant theories.

1.2 Field Theory

Field theory usually refers to a construction of the dynamics of a field, i.e. a specification of how a field changes with respect to other components of the field. Usually this is done by writing a Lagrangian or Hamiltonian of the field, and treating it as the classical mechanics (or quantum mechanics) of a system with an infinite number of degrees of freedom. The resulting field theories are referred to as classical or quantum field theories.

In modern physics, the most often studied are those that model the four fundamental forces.

1.2.1 Classical Fields

There are several examples of classical fields. The dynamics of a classical field are usually specified by the Lagrangian density in terms of the field components; the dynamics can be obtained by using the action principle.

Michael Faraday first realized the importance of a field as a physical object, during his investigations into magnetism. He realized that electric and magnetic fields are not only fields of force which dictate the motion of particles, but also have an independent physical reality because they carry energy.

these ideas eventually led to the creation, by James Clerk Maxwell, of the first unified field theory in physics with the introduction of equations for the electromagnetic field. The modern version of these equations are called Maxwell's equations. At the end of the 19th century, the electromagnetic field was understood as a collection of two vector fields in space. Nowadays, one recognize this as a single antisymmetric 2nd-rank tensor field in spacetime.

Einstein's theory of gravity, called general relativity, is another example of a field theory. Here the principal field is the metric, a symmetric 2nd-rank tensor field in spacetime.

1.2.2 Quantum Fields

It is now believed that quantum mechanics should underlie all physical phenomena, so that a classical field theory should, at least in principle, permit a recasting in quantum mechanical

terms; success yields the corresponding quantum field theory. For example, quantizing classical electrodynamics gives quantum electrodynamics. Quantum electrodynamics is arguably the most successful scientific theory; experimental data confirm its predictions to a higher precision (to more significant digits) than any other theory. The two other fundamental quantum field theories are quantum chromodynamics and the electroweak theory. these three quantum field theories can all be derived as special cases of the so-called standard model of particle physics. General relativity, the classical field theory of gravity, has yet to be successfully quantized.

Classical field theories remain useful wherever quantum properties do not arise, and can be active areas of research. Elasticity of materials, fluid dynamics and Maxwell's equations are cases in point.

1.2.3 Continuous Random Fields

Classical fields as above, such as the electromagnetic field, are usually infinitely differentiable functions, but they are in any case almost always twice differentiable. In contrast, generalized

functions are not continuous. When dealing carefully with classical fields at finite temperature, the mathematical methods of continuous random fields have to be used, because a thermally fluctuating classical field is nowhere differentiable. Random fields are indexed sets of random variables; a continuous random field is a random field that has a set of functions as its index set. In particular, it is often mathematically convenient to take a continuous random field to have a Schwartz space of functions as its index set, in which case the continuous random field is a tempered distribution.

As a (very) rough way to think about continuous random fields, we can think of it as an ordinary function that is $\pm\infty$ almost everywhere, but when we take a weighted average of all the infinities over any finite region, we get a finite result. The infinities are not well-defined, the last sentence is nonsense to a mathematician, but the finite values can be associated with the functions we supposedly used as the weight functions to get the finite values, and that can be well-defined. We can define a continuous random field well enough as a linear map from a space of functions into the real numbers.

1.3 Constrained Systems

Singular Lagrangian systems represent a special case of a more general dynamics called constrained systems [2]. A general feature of constrained system is characterized by the existence of constraints for its classical configurations. The constraints also place restrictions on the possible choice of boundary conditions for the canonical coordinates. Moreover, the standard quantization method can not be applied directly to singular Lagrangian theories.

Most of physicists believe that this distinction is quite important not only in the classical theories but carries through in the quantum mechanics [27].

In the case of the unconstrained systems, the Hamilton-Jacobi theory provides a bridge between classical and quantum mechanics.

The dynamics of the physical system is encoded in the Lagrangian, a function of positions and velocities of all degrees of freedoms, which comprise the system [28]. To extract the dynamics one, consider paths in the configuration space. For a given path one calculates the position and velocities at each time and also the value of the Lagrangian. The Lagrangian

formulation of classical physics requires the configuration space formed by n generalized coordinates q_i , n generalized velocities \dot{q}_i and parameter τ , defined as

$$L \equiv L(q_i, \dot{q}_i; \tau), \quad i = 1, \dots, n. \quad (1.1)$$

where τ is a parameter which henceforth will be the time on which the coordinates q_i depend.

For a system characterized by this Lagrangian, the action which is a function of path in configuration space reads as

$$S = \int L(q_i, \dot{q}_i; \tau) dt. \quad (1.2)$$

The action principle asserts that the path which satisfies the classical equation is the one which brings the action to extremes

$$\begin{aligned} \delta S &= \delta \int L(q_i, \dot{q}_i; \tau) dt. \\ &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial L}{\partial \tau} \delta \tau \right) dt. \end{aligned} \quad (1.3)$$

In deriving (1.3), it was assumed that \dot{q}_i is dependent of q_i , so that $\delta \dot{q}_i = \frac{d}{d\tau} \delta q_i$. Imposing $\delta S = 0$, we obtain the Euler-Lagrange equations of motion

$$\frac{\partial L}{\partial q_i} - \frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0. \quad (1.4)$$

So, the Lagrangian equations are of second order.

To go over the Hamiltonian formalism, defining a generalized

momentum p_i conjugate to q_i as [28,29]

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad (1.5)$$

then the momentum is a function of q_j and \dot{q}_j such that,

$$p_i = p_i(q_j, \dot{q}_j) \quad j = 1, \dots, n. \quad (1.6)$$

The canonical Hamiltonian H_0 is defined by

$$H_0 = \sum_{i=1}^n \dot{q}_i p_i - L. \quad (1.7)$$

Consider the differential of the Lagrange function (1.1) and using eqs. (1.4), (1.5) and (1.7), then we read off the Hamilton's equations of motion as

$$\dot{q}_i = \frac{\partial H_0}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H_0}{\partial q_i}. \quad (1.8)$$

It is standard national practice to define the poisson bracket of two functions f and g on phase space by [20]

$$\{f, g\} = -(-1)^{n_f n_g} \sum_{i=1}^n \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right), \quad (1.9)$$

where

$$n_A = \begin{cases} 0 & \text{if } A \text{ even} \\ 1 & \text{if } A \text{ odd} \end{cases}$$

.

Thus the Hamilton's equation may be written as

$$\dot{q}_i = \{q_i, H_0\}, \quad \dot{p}_i = \{p_i, H_0\}. \quad (1.10)$$

So, the time evolution of any function of positions and momenta is given by

$$\frac{dF}{dt} = \{F, H_0\} + \frac{\partial F}{\partial t}. \quad (1.11)$$

In order to characterize the constrained systems; one evaluates the time derivative of the momentum as

$$\frac{dp_i}{dt} = \frac{\partial p_i}{\partial q_j} \dot{q}_j + \frac{\partial p_i}{\partial \dot{q}_j} \ddot{q}_j. \quad (1.12)$$

Then by using the definition (1.5) and the Lagrange equation of motion (1.4), we get

$$\frac{\partial L}{\partial q_i} - \frac{dp_i}{dt} = 0. \quad (1.13)$$

Using (1.12) and (1.13), we have

$$\frac{\partial L}{\partial q_i} = \frac{\partial p_i}{\partial q_j} \dot{q}_j + \frac{\partial p_i}{\partial \dot{q}_j} \ddot{q}_j. \quad (1.14)$$

$$\frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} \dot{q}_j - \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \ddot{q}_j = 0. \quad (1.15)$$

Defining Hess matrix elements A_{ij} of second derivatives of the Lagrangian with respect to velocities as

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad (1.16)$$

so we can solve \ddot{q}_j as

$$\ddot{q}_j = A_{ij}^{-1} \left[\frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} \dot{q}_j \right]. \quad (1.17)$$

A valid phase space is formed if the rank of the Hess matrix is n . Systems, which posses this property, are called regular and their treatments are found in a standard mechanics books. Systems, which have the rank less than n are called singular systems. Thus, by definition we have [2]

$$Hessian = det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) \begin{cases} \neq 0 & \text{regular system,} \\ = 0 & \text{singular system.} \end{cases} \quad (1.18)$$

To clarify the situation of singular systems, it can be investigated by two different approach of quantization.

1.4 Dirac Approach

The standard quantization methods can't be applied directly to the singular Lagrangian theories. However, the basic idea of the classical treatment and the quantization of such systems were presented along time by Dirac [1,2]. And is now widely used in investigating the theoretical models in a contemporary elementary particle physics and applied in high energy physics, especially in the gauge theories [12].

The presence of constraints in such theories makes one careful on applying Dirac's method, especially when first-class constraints arise. This is because the first-class constraints are

generators of gauge transformation which lead to the gauge freedom [21].

Let us consider a system which is described by the Lagrangian (1.1) such that the rank of the Hess matrix is $(n - r)$, $r < n$.

The singular system characterized by the fact that all velocities \dot{q}_i are not uniquely determined in terms of the coordinates and momenta only. In other words, not all momenta are independent, and there must exist a certain set of relations among them, of the form

$$\phi_m(p_i, q_i) = 0, \quad (1.19)$$

The q 's and the p 's are the dynamical variables of the Hamiltonian theory. They are connected by the relations (1.19) which are called primary constraints of the Hamiltonian formalism.

Since the rank of the Hess matrix is $(n - r)$, the momenta components will be functionally dependent. The first $(n - r)$ equations of (1.5) can be solved for the $(n - r)$ components of \dot{q}_i in terms of q_i as well as the first $(n - r)$ components of p_i and the last r components of \dot{q}_i .

In other words

$$\dot{q}_a = \dot{q}_a(q_i, p_b, \dot{q}_\mu) \equiv \omega_a, \quad (1.20)$$

$$a, b = 1, \dots, n - r, \quad \mu = 1, \dots, r, \quad i = 1, \dots, n.$$

If these expressions for the \dot{q}_a are substituted into the last r equations of (1.5), the resulting equations will yield r relations of the form

$$p_\mu = \left. \frac{\partial L}{\partial \dot{q}_\mu} \right|_{\dot{q}_a = \omega_a} \equiv -H_\mu(q_i, p_a, \dot{q}_\nu). \quad (1.21)$$

These relations indicate that the generalized momenta p_μ are dependent of p_a , which is natural result of the singular nature of the Lagrangian. Eq (1.21) can be written in the form

$$H'_\mu(q_i, p_a, \dot{q}_\nu) \equiv p_\mu + H_\mu \approx 0, \quad (1.22)$$

which are called primary constraints [1,2].

Now the usual Hamiltonian H_0 for any dynamical system is defined as

$$H_0(p_i, q_i) = p_i \dot{q}_i - L \quad (1.23)$$

(Here the Einstein summation rule is used which is a convention when repeated indices are implicitly summed over).

H_0 will not be uniquely determined, since we may add to it any linear combinations of the primary constraints H'_μ 's which are zero, so that the total Hamiltonian is [2, 3, 7]

$$H_T = H_0 + \lambda_\mu H'_\mu \quad (1.24)$$

where $\lambda_\mu(q, p)$ being some unknown coefficients, they are simply Lagrange's undetermined multipliers. Making use of the Poisson brackets, one can write the total time derivative of any function $g(q, p)$ as

$$\dot{g} \equiv \frac{dg}{d\tau} \approx \{g, H_T\} = \{g, H_0\} + \lambda_\mu \{g, H'_\mu\} \quad (1.25)$$

where Dirac's symbol (\approx) for weak equality has been used in the sense that one can't consider $H'_\mu = 0$ identically before working out the Poisson brackets. Thus the equations of motion can be written as

$$\dot{q}_i \approx \{q_i, H_T\} = \{q_i, H_0\} + \lambda_\mu \{q_i, H'_\mu\}, \quad (1.26)$$

$$\dot{p}_i \approx \{p_i, H_T\} = \{p_i, H_0\} + \lambda_\mu \{p_i, H'_\mu\}, \quad (1.27)$$

subject to the so-called consistency conditions. This means that the total time derivative of the primary constraints should be zero;

$$\begin{aligned} \dot{H}'_\mu &\equiv \frac{dH'_\mu}{d\tau} \approx \{H'_\mu, H_T\} \\ &= \{H'_\mu, H_0\} + \lambda_\nu \{H'_\mu, H'_\nu\} \approx 0, \quad \mu, \nu = 1, \dots, r. \end{aligned} \quad (1.28)$$

These equations may be reduced to $0 = 0$, where it is identically satisfied as a result of primary constraints, else they

will be lead to new conditions which are called secondary constraints. Repeating this procedure as many times as needed, one arrives at a final set of constraints or/and specifies some of λ_μ . Such constraints are classified into two types, a) First-class constraints which have vanishing Poisson brackets with all other constraints. b) Second-class constraints which have non-vanishing Poisson brackets. The second-class constraints could be used to eliminate conjugated pairs of the p 's and q 's from the theory by expressing them as functions of the remaining p 's and q 's. The total Hamiltonian for the remaining variable is then the canonical Hamiltonian plus the primary constraints H'_μ of the first type as in eq. (1.24), where H'_μ are all the independent remaining first-class constraints.

The first-class constraints are the generators of the gauge transformations. This will lead to the gauge freedom. Besides, λ_μ are still undetermined. To remove this arbitrariness, one has to impose external gauge constraints for each first-class constraints. Such a gauge fixing,

$$\chi = 0, \tag{1.29}$$

which is a set of constraints independent of H'_μ and equal in number to all first-class constraints H'_μ . This choice makes the

whole set of constraints $\{H'_\mu, H'_\nu\}$ to be second-class constraints, with

$$\det\{H'_\mu, H'_\nu\} \neq 0, \quad \mu, \nu = n - r + 1, \dots, n. \quad (1.30)$$

This is a canonical physical gauge if it does not violate the equations of motion [3,21].

Fixing any gauge is not an easy task, since we fix it by hand and there is no basic rule to select it, specially in the general relativity.

1.5 Hamilton-Jacobi Approach

Now we would like to approach the constrained systems by Hamilton-Jacobi treatment [15-17], and demonstrate the fact that the gauge-fixing problem is solved naturally.

Güler [15,16] has developed a completely different method to investigate singular systems. He started with the Hess matrix given in equation (1.16) of rank $(n - r)$. Then r of the momenta are dependent. The equivalent Lagrangian method [21] is used to obtain the set of Hamilton-Jacobi Partial Differential Equations (HJPDE). The generalized momenta corresponding

to generalized coordinates φ_i are defined as

$$\pi_a = \frac{\overrightarrow{\delta} \mathcal{L}}{\delta(\partial_\mu \varphi_a)}, \quad a = 1, 2, \dots, n-r, \quad (1.31)$$

$$\pi_j = \frac{\overrightarrow{\delta} \mathcal{L}}{\delta(\partial_\mu \varphi_j)}, \quad j = n-r+1, \dots, n, \quad (1.32)$$

where φ_i are divided into two sets φ_a and x_j . Since the rank of Hess matrix is $(n-r)$, one may solve eq. (1.31) for $\partial_\mu \varphi_a$ as

$$\partial_\mu \varphi_a = \partial_\mu \varphi_a(\varphi_i, \pi_a, \partial_\mu \varphi_j; \chi_\mu) \equiv \omega_a, \quad (1.33)$$

Substituting eq. (1.33), into eq. (1.32), we get

$$\pi_j = \left. \frac{\overrightarrow{\delta} \mathcal{L}}{\delta(\partial_\mu \varphi_j)} \right|_{\partial_\mu \varphi_a = \omega_a} \equiv -\mathcal{H}_j(\varphi_i, \partial_\mu \varphi_\nu, \pi_a; \chi_\mu). \quad (1.34)$$

Relations (1.34) indicate the fact that the generalized momenta π_j are not independent of π_a which is a natural result of the singular nature of the Lagrangian.

The canonical Hamiltonian H_o is defined as

$$\mathcal{H}_0 = -\mathcal{L}(\varphi_i, \partial_\mu \varphi_\nu, \partial_\mu \varphi_a \equiv \omega_a, \chi_\mu) + \pi_a \omega_a + \pi_j \partial_\mu \varphi_j \Big|_{\pi_\nu = -\mathcal{H}_\nu}. \quad (1.35)$$

The set of the Hamilton-Jacobi Partial Differential Equations (HJPDE) is expressed as

$$\mathcal{H}'_0 \left(\tau, \varphi_\nu, \varphi_a, \pi_i = \frac{\overrightarrow{\delta} S}{\delta \varphi_i}, \pi_0 = \frac{\overrightarrow{\delta} S}{\delta \chi_\mu} \right) = 0, \quad (1.36)$$

$$\mathcal{H}'_\mu \left(\tau, \varphi_\nu, \varphi_a, \pi_i = \frac{\overrightarrow{\delta} S}{\delta \varphi_i}, \pi_0 = \frac{\overrightarrow{\delta} S}{\delta \chi_\mu} \right) = 0, \quad (1.37)$$

where S being the action.

Eqs (1.36) and (1.37) may be expressed in a compact form as

$$\mathcal{H}'_\alpha \left(\tau, \varphi_\nu, \varphi_a, \pi_i = \frac{\overrightarrow{\delta} S}{\delta \varphi_i}, \pi_0 = \frac{\overrightarrow{\delta} S}{\delta \chi_\alpha} \right) = 0, \quad (1.38)$$

$$\alpha = 0, n - r + 1, \dots, n.$$

where

$$\mathcal{H}'_0 = \pi_0 + \mathcal{H}_0 = 0, \quad \mathcal{H}'_\mu = \pi_\mu + \mathcal{H}_\mu = 0. \quad (1.39)$$

Here \mathcal{H}'_0 can be interpreted as the generator of time evolution while \mathcal{H}'_j are the generators of gauge transformation.

The fundamental equations of the equivalent Lagrangian method are

$$\pi_0 = \frac{\overrightarrow{\delta} S}{\delta \chi_\mu} \equiv -\mathcal{H}_0(\varphi_i, \delta_\mu \varphi_\nu, \pi_a; \chi_\mu), \quad \pi_a = \frac{\overrightarrow{\delta} S}{\delta \varphi_a},$$

$$\pi_j = \frac{\overrightarrow{\delta} S}{\delta \varphi_j} \equiv -\mathcal{H}_j, \quad (1.40)$$

with $\varphi_0 = \chi_\mu$.

Now the equations of motion are obtained as total differential

equations in many variables as follows:

$$d\varphi_r = \frac{\overrightarrow{\delta} \mathcal{H}'_\alpha}{\delta \pi_r} d\chi_\alpha, \quad r = 0, 1, \dots, n, \quad (1.41)$$

$$d\pi_a = -\frac{\overrightarrow{\delta} \mathcal{H}'_\alpha}{\delta \varphi_a} d\chi_\alpha, \quad a = 1, \dots, n-r, \quad (1.42)$$

$$d\pi_\mu = -\frac{\overrightarrow{\delta} \mathcal{H}'_\alpha}{\delta \varphi_\mu} d\chi_\alpha, \quad \mu = n-r+1, \dots, n, \quad (1.43)$$

$$dZ = \left(-\mathcal{H}_\alpha + \pi_a \frac{\overrightarrow{\delta} \mathcal{H}'_\alpha}{\delta \pi_a} \right) d\chi_\alpha, \quad (1.44)$$

$$\alpha = 0, n-r+1, \dots, n,$$

where $Z = S(\chi_\alpha, \varphi_a)$. These equations are integrable if and only if

$$d\mathcal{H}'_0 = 0, \quad (1.45)$$

$$d\mathcal{H}'_\beta = 0, \quad \beta = 1, 2, \dots, r. \quad (1.46)$$

If conditions (1.45) and (1.46) are not satisfied identically, then we consider them as new constraints and again we examine the variations of them. Thus, repeating this procedure one may obtain a set of conditions such that all variations vanish.

The investigation of the integrability conditions [32,33] can be also done by using the operator method, where the linear operators X_α corresponding to the set (1.41-1.43) are defined

as

$$X_\alpha f(\chi_\beta, \varphi_a, \pi_a, z) = \frac{\delta f}{\delta \chi_\alpha} + \frac{\delta \mathcal{H}'_\alpha}{\delta \pi_a} \frac{\delta f}{\delta \varphi_a} - \frac{\delta \mathcal{H}'_\alpha}{\delta \varphi_a} \frac{\delta f}{\delta \pi_a} + \left(-\mathcal{H}_\alpha + \pi_a \frac{\delta \mathcal{H}'_\alpha}{\delta \pi_a} \right) \frac{\delta f}{\delta z}. \quad (1.47)$$

The system is integrable, if the bracket relations

$$[X_\alpha, X_\beta]f = (X_\alpha X_\beta - X_\beta X_\alpha)f = C_{\alpha\beta}^\gamma X_\gamma f; \quad (1.48)$$

$$\forall \alpha, \beta, \gamma = 0, n-r+1, \dots, n,$$

are hold. If the relations (1.41-1.43) are not satisfied identically, we add the bracket relations, which cannot be expressed in this form as new operators. So the numbers of independent operators are increased, and a new complete system can be obtained. Then the new operators can be written in the Jacobi form, and we find the corresponding integrable system of the total differential equations.

Chapter 2

Applications

In this chapter we will consider three applications on both Dirac and Hamilton-Jacobi approach.

2.1 Preliminaries

The dynamics of the continuous systems is described by a function $Q(x)$ of space-time, rather than functions of time $q_i(t)$ in discrete systems. The discrete label i is replaced by the continuous label $x \equiv (ct, \bar{x})$. Further, in continuous systems the function of coordinate $f(q)$ becomes a functional $F[Q]$ of fields.

The most general form of the Lagrangian in the field theory is the functional of fields as well as their time and space

derivatives, that is [29],

$$L = \int \mathcal{L} d^3x, \quad (2.1)$$

where

$$\mathcal{L} = \mathcal{L}(Q_r, \partial^\mu Q_r), \quad r = 1, 2, 3, \quad \mu = 0, 1, 2, 3. \quad (2.2)$$

is the corresponding Lagrangian density with

$$\partial^\mu Q_r \equiv \frac{\partial Q_r}{\partial x_\mu}. \quad (2.3)$$

At this point we must decide on a metric convention for treating covariant and contravariant vectors in four-dimensional space-time. The relation between the covariant vector A_μ and its contravariant partner A^μ is defined as [29,30]

$$A_\mu = g_{\mu\nu} A^\nu \quad \mu, \nu = 0, 1, 2, 3. \quad (2.4)$$

where its inverse is defined as

$$A^\mu = g^{\mu\nu} A_\nu, \quad (2.5)$$

where $g_{\mu\nu}$ is the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (2.6)$$

With this definition it follows that, if a contravariant 4-vector has the components A^0, A^1, A^2, A^3 , its covariant partner will have the components $A_0 = A^0, A_1 = -A^1, A_2 = -A^2, A_3 = -A^3$. This can be written concisely as

$$A^\mu = (A^0, \vec{A}); \quad A_\mu = (A_0, -\vec{A}). \quad (2.7)$$

These are transformed under the Lorentz transformation according to the rule

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu; \quad A'_\mu = \frac{\partial x_\nu}{\partial x'_\mu} A_\nu. \quad (2.8)$$

The differential with respect to a covariant component gives a contravariant vector operator. We therefore employ the notation

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right); \quad (2.9)$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla} \right).$$

2.2 Lagrangian with Fermionic and Scalar Field

2.2.1 Fermions

Dirac considered the problem with the probability interpretation of Klein-Gordon equation, as a second order in time derivatives. He therefore set out to find a relativistic wave equation with only one time derivative. The requirement that the form of the equation be unchanged under Lorentz transformations, which mix up $\partial/\partial t$ and ∇ , means that the equation must be first order in spatial derivatives as well. Hence Dirac proposed a relativistic free particle wave equation

$$i\frac{\partial}{\partial t}\psi = -i\boldsymbol{\alpha} \cdot \nabla\psi + \beta m\psi. \quad (2.10)$$

Therefore clearly has to be something rather special about the objects α , β and ψ in order that Lorentz covariance be preserved. In fact, α and β are 4×4 Hermitian matrices, and ψ is an object with four components called a *spinor*.

Now, we want a wave equation with plane solutions which satisfy the relativistic energy-momentum relation $E^2 = \mathbf{P}^2 c^2 + m^2 c^4$. To see how this property emerges from the Dirac equa-

tion, we act on both sides of (2.10) with $(i\partial_t - i\alpha \cdot \nabla)$, to obtain

$$\left(i\frac{\partial}{\partial t} - i\alpha \cdot \nabla\right)\left(i\frac{\partial}{\partial t} + i\alpha \cdot \nabla\right)\psi = m\left(i\frac{\partial}{\partial t} - i\alpha \cdot \nabla\right)\beta\psi. \quad (2.11)$$

At this point, it is more convenient to switch to index notation for the 3-vector α and ∇ , noting that $\alpha \cdot \nabla = \alpha^i \partial_i$. We expand out the brackets on the left hand side of (2.11), and then add and subtract the quantity $-i\beta\alpha^i \partial_i$ on the right hand side. The result is

$$\left(-\frac{\partial^2}{\partial t^2} + \alpha^i \alpha^j \partial_i \partial_j\right)\psi = m\left[\left(i\beta\frac{\partial\psi}{\partial t} - i\beta\alpha^i \partial_i\psi\right) + i\beta\alpha^i \partial_i\psi + i\alpha^i \beta \partial_i\psi\right]. \quad (2.12)$$

It is now useful to define the *anticommutator*, represented by curly brackets:

$$\{A, B\} = -(-1)^{n_A n_B} \{B, A\}. \quad (2.13)$$

(Sometimes you will also see the anticommutator, written as $[A, B]_+$). Substitute from Eq. (2.10), we rewrite the left hand side as $(-\partial_t^2 + \frac{1}{2}\alpha^i \alpha^j \partial_i \partial_j + \frac{1}{2}\alpha^j \alpha^i \partial_j \partial_i)\psi$, which we are entitled to do by renaming the indices on α and ∂ , to obtain

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{1}{2}\{\alpha^i, \alpha^j\}\partial_i \partial_j\right)\psi = m\beta[-i\alpha^i \partial_i + \beta m]\psi - im\beta\alpha^i \partial_i\psi. \quad (2.14)$$

In order to reproduce the relativistic relation between energy and momentum, we must end up with an equation like the

Klein-Gordon equation. If α^i and β satisfy

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij}, \quad \{\beta, \alpha^i\} = 0, \quad \beta^2 = 1, \quad (2.15)$$

where
$$\delta^{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases},$$

and 1 is the 4×4 identity matrix, ψ satisfies the equation

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 \right) \psi = m^2 \psi. \quad (2.16)$$

Hence, each of the four components of ψ satisfy the Klein-Gordon equation.

We can make the Dirac equation more explicitly relativistic by defining four new 4×4 matrices:

$$\gamma^0 = \beta, \quad \gamma^i = \beta \alpha^i. \quad (2.17)$$

If we multiply both sides of the Dirac equation (2.10) by β , we obtain

$$(i\gamma^\mu \partial_\mu - m) \psi = 0. \quad (2.18)$$

The conditions (2.15) on α^i and β are neatly unified into the matrix equation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} 1. \quad (2.19)$$

where $\eta^{\mu\nu}$ is a metric of flat space time.

$$\eta^{00} = +1, \quad \eta^{ii} = -1 \quad \forall i = 1, 2, 3, \quad \eta^{\mu\nu} = 0, \quad \forall \mu \neq \nu.$$

2.2.2 Hamilton-Jacobi Approach to Lagrangian with Fermionic and Scalar Field

Consider a Lagrangian containing elements of a fermionic field ψ with mass m and a scalar field σ with mass M given by

$$L = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) + \frac{1}{2} \partial_\mu \sigma(x) \partial^\mu \sigma(x) - \frac{1}{2} M^2 \sigma^2(x),$$

$$\mu = 0, 1, 2, 3. \quad (2.20)$$

We are adopting the Minkowski metric $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$.

The Lagrangian (2.20) is singular, since the rank of the Hess matrix (1.16) is one. The generalized momenta (1.31,1.32) can be written as

$$p = \frac{\vec{\partial} L}{\partial \dot{\sigma}} = \partial^0 \sigma = \omega, \quad (2.21)$$

$$p_\psi = \frac{\vec{\partial} L}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0 = -H_\psi, \quad (2.22)$$

$$p_{\bar{\psi}} = \frac{\vec{\partial} L}{\partial \dot{\bar{\psi}}} = 0 = -H_{\bar{\psi}}, \quad (2.23)$$

where we must call attention to the necessity of being careful with the spinor indexes. Considering, as usual ψ as a column vector and $\bar{\psi}$ as a row vector implies that p_ψ will be a row vector while $p_{\bar{\psi}}$ will be a column vector.

The Hamiltonian density H_0 is given as

$$H_0 = -L + \omega p + \partial_0 \psi p_\psi \Big|_{p_\psi = -H_\psi} + \partial_0 \bar{\psi} p_{\bar{\psi}} \Big|_{p_{\bar{\psi}} = -H_{\bar{\psi}}}, \quad (2.24)$$

or

$$H_0 = \frac{1}{2} p^2 - \bar{\psi} (i \gamma^a \partial_a \psi - m \psi) - \frac{1}{2} (\partial_a \sigma \partial^a \sigma - M^2 \sigma^2), \quad a = 1, 2, 3. \quad (2.25)$$

The set of Hamilton-Jacobi Partial differential equations (HJPDE) (1.38) read as

$$H'_0 = p_0 + H_0 = p_0 + \frac{1}{2} p^2 - \bar{\psi} (i \gamma^a \partial_a \psi - m \psi) - \frac{1}{2} (\partial_a \sigma \partial^a \sigma - M^2 \sigma^2), \quad (2.26)$$

$$H'_\psi = p_\psi + H_\psi = p_\psi - i \bar{\psi} \gamma^0 = 0, \quad (2.27)$$

$$H'_{\bar{\psi}} = p_{\bar{\psi}} + H_{\bar{\psi}} = p_{\bar{\psi}} = 0. \quad (2.28)$$

Therefor, the total differential equations for the characteristic (1.41), (1.42) and (1.43) are

$$d\sigma = p d\tau, \quad (2.29)$$

$$dp = -M^2 \sigma d\tau, \quad (2.30)$$

$$dp_\psi = -\bar{\psi} (i \overleftarrow{\partial}_a \gamma^a + m) d\tau, \quad (2.31)$$

$$dp_{\bar{\psi}} = (i \gamma^a \partial_a - m) \psi d\tau + i \gamma^0 d\psi \quad (2.32)$$

The integrability conditions ($dH'_\alpha = 0$) imply that the variation of the constraints H'_ψ and $H'_{\bar{\psi}}$ should be identically zero, that is

$$dH'_\psi = dp_\psi - i d\bar{\psi} \gamma^0 = 0, \quad (2.33)$$

$$dH'_{\bar{\psi}} = dp_{\bar{\psi}} = 0. \quad (2.34)$$

Substituting from eqs. (2.31) and (1.32) into eqs. (2.33) and (2.34), respectively we get the following equations of motion:

$$\dot{\sigma} = p. \quad (2.35)$$

$$\bar{\psi}(i \overleftarrow{\partial}_\mu \gamma^\mu + m) = 0, \quad (2.36)$$

$$(i \gamma^\mu \partial_\mu - m)\psi = 0. \quad (2.37)$$

In addition, from eqs. (2.30-2.32), we get another set of equations of motion:

$$\dot{p} = -M^2 \sigma, \quad (2.38)$$

$$\dot{p}_\psi = -\bar{\psi}(i \overleftarrow{\partial}_a \gamma^a + m), \quad (2.39)$$

$$\dot{p}_{\bar{\psi}} = 0. \quad (2.40)$$

Differentiate eq.(2.35) with respect to time and making use of eq.(2.38), we obtain

$$\ddot{\sigma} + M^2 \sigma = 0. \quad (2.41)$$

In the following section the same system will be discussed using Dirac's approach.

2.2.3 Dirac's Approach to Lagrangian with Fermionic and Scalar Field

The total Hamiltonian is given as

$$H_T = H_0 + \lambda_\psi H'_\psi + \lambda_{\bar{\psi}} H'_{\bar{\psi}}, \quad (2.42)$$

or

$$\begin{aligned} H_T = \frac{1}{2}p^2 - \bar{\psi}(i\gamma^a\partial_a\psi - m\psi) - \frac{1}{2}(\partial_a\sigma\partial^a\sigma - M^2\sigma^2) \\ + \lambda_\psi(p_\psi - i\bar{\psi}\gamma^0) + \lambda_{\bar{\psi}}p_{\bar{\psi}}. \end{aligned} \quad (2.43)$$

According to Dirac's method, the time derivative of the primary constraints should be zero, that is

$$\dot{H}'_\psi = \{H'_\psi, H_T\} = -(i\partial_a\bar{\psi}\gamma^a + m\bar{\psi}) - i\lambda_{\bar{\psi}}\gamma^0 \approx 0, \quad (2.44)$$

$$\dot{H}'_{\bar{\psi}} = \{H'_{\bar{\psi}}, H_T\} = (i\gamma^a\partial_a - m)\psi + i\gamma^0\lambda_\psi \approx 0. \quad (2.45)$$

Eqs. (2.44) and (2.45) fix the multipliers $\lambda_{\bar{\psi}}$ and λ_ψ , respectively as

$$i\lambda_{\bar{\psi}}\gamma^0 = -(i\partial_a\bar{\psi}\gamma^a + m\bar{\psi}), \quad (2.46)$$

$$i\gamma^0\lambda_\psi = -(i\gamma^a\partial_a - m)\psi. \quad (2.47)$$

Multiplying eq.(2.46) from the right and eq.(2.47) from the left by $-i\gamma^0$, we obtain

$$\lambda_{\bar{\psi}} = -\partial_a\bar{\psi}\gamma^a\gamma^0 + im\bar{\psi}\gamma^0, \quad (2.48)$$

$$\lambda_\psi = -\gamma^0(\gamma^a \partial_a + im)\psi. \quad (2.49)$$

There are no secondary constraints. Taking suitable linear combinations of constraints, one has to find all numbers of second-class ones, there are

$$\Phi_1 = H'_\psi = p_\psi - i\bar{\psi}\gamma^0, \quad (2.50)$$

and

$$\Phi_2 = H'_{\bar{\psi}} = p_{\bar{\psi}}. \quad (2.51)$$

The total Hamiltonian is vanishing weakly. It can completely be written in terms of second-class constraints as

$$H_T = \frac{1}{2}p^2 - \bar{\psi}(i\gamma^a \partial_a \psi - m\psi) - \frac{1}{2}(\partial_a \sigma \partial^a \sigma - M^2 \sigma^2) + \lambda_\psi \Phi_1 + \lambda_{\bar{\psi}} \Phi_2. \quad (2.52)$$

According to Dirac, the equations of motion read as

$$\dot{\sigma} = \{\sigma, H_T\} = p, \quad (2.53)$$

$$\dot{\psi} = \{\psi, H_T\} = \lambda_\psi, \quad (2.54)$$

$$\dot{\bar{\psi}} = \{\bar{\psi}, H_T\} = \lambda_{\bar{\psi}}, \quad (2.55)$$

$$\dot{p} = \{p, H_T\} = -M^2 \sigma, \quad (2.56)$$

$$\dot{p}_\psi = \{p_\psi, H_T\} = -\bar{\psi}(i\overleftarrow{\partial}_a \gamma^a + m), \quad (2.57)$$

$$\dot{p}_{\bar{\psi}} = \{p_{\bar{\psi}}, H_T\} = (i\gamma^a \partial_a - m)\psi + i\gamma^0 \lambda_\psi. \quad (2.58)$$

Differentiating eq.(2.53) with respect to time, and using eq (4.56), we have

$$\ddot{\sigma} + M^2\sigma = 0, \quad (2.59)$$

same as (2.41). Substituting from eqs.(2.48) and (2.49) into eqs.(2.55) and (2.54), respectively we get the same equations obtained in the previous section

$$\overline{\psi}(i\overleftarrow{\partial}_\mu\gamma^\mu + m) = 0, \quad (2.60)$$

$$(i\gamma^\mu\partial_\mu - m)\psi = 0. \quad (2.61)$$

Substituting from eq.(2.49) into eq. (2.58), we get

$$\dot{p}_{\overline{\psi}} = 0. \quad (2.62)$$

As a comparison between Hamilton-Jacobi method and Dirac's method, one notes that the two methods give the same equations of motion.

2.3 The Scalar Field Coupled to Two Flavours of Fermions Through Yukawa Couplings

Consider one-loop order the self-energy for the scalar field φ with a mass m , coupled to two flavours of fermions with

masses m_1 and m_2 , coupled through Yukawa couplings described by the Lagrangian

$$L = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2 \varphi^2 - \frac{1}{6}\lambda \varphi^3 + \sum_i \bar{\psi}_{(i)}(i\gamma^\mu \partial_\mu - m_i)\psi_{(i)} - g\varphi(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}), \quad \mu = 0, 1, 2, 3, \quad (2.63)$$

where λ is parameter and g constant, φ , $\psi_{(i)}$, and $\bar{\psi}_{(i)}$ are odd ones.

The self-energy for the scalar field in one-loop order splites in two contributions Σ_1 and Σ_2 from a fermion and a scalar loop (in this order $Z \equiv 1$).

$$\Sigma_\varphi(q) \equiv \Sigma_1 + \Sigma_2. \quad (2.64)$$

2.3.1 Hamilton-Jacobi Formulation of The Scalar Field Coupled to Two Flavours of Fermions Through Yukawa Couplings

The Lagrangian (2.63) is singular, since the rank of the Hess matrix (1.16) is one. The generalized momenta (1.31,1.32) take the forms

$$p_\varphi = \frac{\vec{\partial} L}{\partial \dot{\varphi}} = \partial^0 \varphi, \quad (2.65)$$

$$p_{(i)} = \frac{\vec{\partial} L}{\partial \dot{\psi}_{(i)}} = i\bar{\psi}_{(i)}\gamma^0 = -H_{(i)}, \quad i = 1, 2, \quad (2.66)$$

$$\bar{p}_{(i)} = \frac{\vec{\partial} L}{\partial \dot{\bar{\psi}}_{(i)}} = 0 = -\bar{H}_{(i)}. \quad (2.67)$$

Where we must call attention to the necessity of being careful with the spinor indexes. Considering, as usual $\psi_{(i)}$ as a column vector and $\bar{\psi}_{(i)}$ as a row vector implies that $p_{(i)}$ will be a row vector while $\bar{p}_{(i)}$ will be a column vector.

Since the rank of the Hess matrix is one, one may solve (2.65) for $\partial^0 \varphi$ as

$$\partial^0 \varphi = p_\varphi \equiv \omega. \quad (2.68)$$

The usual Hamiltonian H_0 is given as

$$H_0 = -L + \omega p_\varphi + \partial_0 \psi_{(i)} p_{(i)} \Big|_{p_{(i)} = -H_{(i)}} + \partial_0 \bar{\psi}_{(i)} \bar{p}_{(i)} \Big|_{\bar{p}_{(i)} = -\bar{H}_{(i)}}, \quad (2.69)$$

or

$$H_0 = \frac{1}{2}(p_\varphi^2 - \partial_a \varphi \partial^a \varphi) + \frac{1}{2}m^2 \varphi^2 + \frac{1}{6}\lambda \varphi^3 - \bar{\psi}_{(i)}(i\gamma^a \partial_a - m_i)\psi_{(i)} \\ + g\varphi(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}), \quad a = 1, 2, 3. \quad (2.70)$$

The set of Hamilton-Jacobi Partial Differential Equations (HJPDE) (1.38) read as

$$H'_0 = p_0 + H_0 \\ = p_0 + \frac{1}{2}(p_\varphi^2 - \partial_a \varphi \partial^a \varphi) + \frac{1}{2}m^2 \varphi^2 + \frac{1}{6}\lambda \varphi^3 - \bar{\psi}_{(i)}(i\gamma^a \partial_a - m_i)\psi_{(i)} \\ + g\varphi(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}), \quad (2.71)$$

$$H'_{(i)} = p_{(i)} + H_{(i)} = p_{(i)} - i \bar{\psi}_{(i)} \gamma^0 = 0, \quad (2.72)$$

$$\bar{H}'_{(i)} = \bar{p}_{(i)} + \bar{H}_{(i)} = \bar{p}_{(i)} = 0. \quad (2.73)$$

The equations of motion are obtained as total differential equations as follows:

$$\begin{aligned} d\varphi &= \frac{\partial H'_0}{\partial p_\varphi} d\tau + \frac{\partial H'_{(i)}}{\partial p_\varphi} d\psi_{(i)} + \frac{\partial \bar{H}'_{(i)}}{\partial p_\varphi} d\bar{\psi}_{(i)}, \\ &= p_\varphi d\tau, \end{aligned} \quad (2.74)$$

$$\begin{aligned} dp_\varphi &= \frac{\partial H'_0}{\partial \varphi} d\tau + \frac{\partial H'_{(i)}}{\partial \varphi} d\psi_{(i)} + \frac{\partial \bar{H}'_{(i)}}{\partial \varphi} d\bar{\psi}_{(i)}, \\ &= \left[m^2 \varphi + \frac{1}{2} \lambda \varphi^2 + g(\bar{\psi}_{(1)} \psi_{(2)} + \bar{\psi}_{(2)} \psi_{(1)}) \right] d\tau, \end{aligned} \quad (2.75)$$

$$\begin{aligned} dp_{(1)} &= \frac{\partial H'_0}{\partial \psi_{(1)}} d\tau + \frac{\partial H'_{(i)}}{\partial \psi_{(1)}} d\psi_{(i)} + \frac{\partial \bar{H}'_{(i)}}{\partial \psi_{(1)}} d\bar{\psi}_{(i)}, \\ &= \left[\bar{\psi}_{(1)} (i \overleftarrow{\partial}_a \gamma^a + m_1) + g \varphi \bar{\psi}_{(2)} \right] d\tau, \end{aligned} \quad (2.76)$$

$$\begin{aligned} dp_{(2)} &= \frac{\partial H'_0}{\partial \psi_{(2)}} d\tau + \frac{\partial H'_{(i)}}{\partial \psi_{(2)}} d\psi_{(i)} + \frac{\partial \bar{H}'_{(i)}}{\partial \psi_{(2)}} d\bar{\psi}_{(i)}, \\ &= \left[\bar{\psi}_{(2)} (i \overleftarrow{\partial}_a \gamma^a + m_2) + g \varphi \bar{\psi}_{(1)} \right] d\tau, \end{aligned} \quad (2.77)$$

$$\begin{aligned}
d\bar{p}_{(1)} &= \frac{\partial H'_0}{\partial \bar{\psi}_{(1)}} d\tau + \frac{\partial H'_{(i)}}{\partial \bar{\psi}_{(1)}} d\psi_{(i)} + \frac{\partial \bar{H}'_{(i)}}{\partial \bar{\psi}_{(1)}} d\bar{\psi}_{(i)}, \\
&= \left[-(i\gamma^a \partial_a - m_1)\psi_{(1)} + g\varphi\psi_{(2)} \right] d\tau - i\gamma^0 d\psi_{(1)}, \quad (2.78)
\end{aligned}$$

$$\begin{aligned}
d\bar{p}_{(2)} &= \frac{\partial H'_0}{\partial \bar{\psi}_{(2)}} d\tau + \frac{\partial H'_{(i)}}{\partial \bar{\psi}_{(2)}} d\psi_{(i)} + \frac{\partial \bar{H}'_{(i)}}{\partial \bar{\psi}_{(2)}} d\bar{\psi}_{(i)}, \\
&= \left[-(i\gamma^a \partial_a - m_2)\psi_{(2)} + g\varphi\psi_{(1)} \right] d\tau - i\gamma^0 d\psi_{(2)}. \quad (2.79)
\end{aligned}$$

The integrability conditions ($dH'_\alpha = 0$) imply that the variation of the constraints $H'_{(i)}$ and $\bar{H}'_{(i)}$ should be identically zero, that is

$$dH'_{(i)} = dp_{(i)} - i d\bar{\psi}_{(i)} \gamma^0 = 0, \quad (2.80)$$

$$d\bar{H}'_{(i)} = d\bar{p}_{(i)} = 0. \quad (2.81)$$

Substituting from eqs. (2.76) and (2.77) into eq. (2.80), we get

$$i\partial_0 \bar{\psi}_{(1)} \gamma^0 - \bar{\psi}_{(1)} (i\overleftarrow{\partial}_a \gamma^a + m_1) - g\varphi \bar{\psi}_{(2)} = 0, \quad (2.82)$$

$$i\partial_0 \bar{\psi}_{(2)} \gamma^0 - \bar{\psi}_{(2)} (i\overleftarrow{\partial}_a \gamma^a + m_2) - g\varphi \bar{\psi}_{(1)} = 0. \quad (2.83)$$

Substituting from eqs. (2.78) and (2.79) into eq. (2.81), we have

$$(i\gamma^\mu \partial_\mu - m_1)\psi_{(1)} - g \varphi \psi_{(2)} = 0, \quad (2.84)$$

$$(i\gamma^\mu \partial_\mu - m_2)\psi_{(2)} - g \varphi \psi_{(1)} = 0. \quad (2.85)$$

One notes that the integrability conditions are not identically zero, they are added to the set of equations of motion.

From eqs.(2.75-2.77), we get the following equations of motion:

$$\dot{p}_\varphi = m^2 \varphi + \frac{1}{2} \lambda \varphi^2 + g(\bar{\psi}_{(1)} \psi_{(2)} + \bar{\psi}_{(2)} \psi_{(1)}), \quad (2.86)$$

$$\dot{p}_{(1)} = \bar{\psi}_{(1)}(i\overleftarrow{\partial}_a \gamma^a + m_1) + g \varphi \bar{\psi}_{(2)}, \quad (2.87)$$

$$\dot{p}_{(2)} = \bar{\psi}_{(2)}(i\overleftarrow{\partial}_a \gamma^a + m_2) + g \varphi \bar{\psi}_{(1)}. \quad (2.88)$$

Substituting from eqs(2.84) and (2.85) into (2.78) and (2.79), we get

$$\dot{\bar{p}}_{(i)} = 0, \quad i = 1, 2. \quad (2.89)$$

Differentiate eq.(2.74) with respect to time, and making use of (2.86), we have

$$\ddot{\varphi} - m^2 \varphi - \frac{1}{2} \lambda \varphi^2 - g(\bar{\psi}_{(1)} \psi_{(2)} + \bar{\psi}_{(2)} \psi_{(1)}) = 0. \quad (2.90)$$

2.3.2 Dirac's Formulation of The Scalar Field Coupled to Two FLavours of Fermions Through Yukawa Couplings

The problem is going now to be tackled by using Dirac's method.

The total Hamiltonian is given as

$$H_T = H_0 + \lambda_{(i)} H'_{(i)} + \bar{\lambda}_{(i)} \bar{H}'_{(i)}, \quad (2.91)$$

or

$$\begin{aligned} H_T = & \frac{1}{2}(p^2_\varphi - \partial_a \varphi \partial^a \varphi) + \frac{1}{2}m^2 \varphi^2 + \frac{1}{6}\lambda \varphi^3 - \bar{\psi}_{(i)}(i\gamma^a \partial_a - m_i)\psi_{(i)} \\ & + g\varphi(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}) + \lambda_{(i)}(p_{(i)} - i\bar{\psi}_{(i)}\gamma^0) + \bar{\lambda}_{(i)}\bar{p}_{(i)}. \end{aligned} \quad (2.92)$$

According to Dirac's method, the time derivative of the primary constraints should be zero, that is

$$\dot{H}'_{(1)} = \{H'_{(1)}, H_T\} = \bar{\psi}_{(1)}(i\overleftarrow{\partial}_a \gamma^a + m_1) + g\varphi \bar{\psi}_{(2)} - i\bar{\lambda}_{(1)}\gamma^0 \approx 0, \quad (2.93)$$

$$\dot{H}'_{(2)} = \{H'_{(2)}, H_T\} = \bar{\psi}_{(2)}(i\overleftarrow{\partial}_a \gamma^a + m_2) + g\varphi \bar{\psi}_{(1)} - i\bar{\lambda}_{(2)}\gamma^0 \approx 0, \quad (2.94)$$

$$\dot{\bar{H}}'_{(1)} = \{\bar{H}'_{(1)}, H_T\} = -(i\gamma^a \partial_a - m_1)\psi_{(1)} + g\varphi \psi_{(2)} - i\gamma^0 \lambda_{(1)} \approx 0, \quad (2.95)$$

$$\dot{\bar{H}}'_{(2)} = \{\bar{H}'_{(2)}, H_T\} = -(i\gamma^a \partial_a - m_2)\psi_{(2)} + g\varphi \psi_{(1)} - i\gamma^0 \lambda_{(2)} \approx 0. \quad (2.96)$$

Eqs. (2.93-2.96) fix the multipliers $\bar{\lambda}_{(1)}$, $\bar{\lambda}_{(2)}$, $\lambda_{(1)}$, and $\lambda_{(2)}$ respectively as

$$i\bar{\lambda}_{(1)}\gamma^0 = \bar{\psi}_{(1)}(i\overleftarrow{\partial}_a\gamma^a + m_1) + g\varphi\bar{\psi}_{(2)}, \quad (2.97)$$

$$i\bar{\lambda}_{(2)}\gamma^0 = \bar{\psi}_{(2)}(i\overleftarrow{\partial}_a\gamma^a + m_2) + g\varphi\bar{\psi}_{(1)}, \quad (2.98)$$

$$i\gamma^0\lambda_{(1)} = -(i\gamma^a\partial_a - m_1)\psi_{(1)} + g\varphi\psi_{(2)}, \quad (2.99)$$

$$i\gamma^0\lambda_{(2)} = -(i\gamma^a\partial_a - m_2)\psi_{(2)} + g\varphi\psi_{(1)}. \quad (2.100)$$

Multiplying eqs.(2.97) and (2.98) from the right and eqs.(2.99) and (2.100) from the left by $-i\gamma^0$, we obtain

$$\bar{\lambda}_{(1)} = \bar{\psi}_{(1)}(\overleftarrow{\partial}_a\gamma^a - im_1)\gamma^0 - ig\varphi\bar{\psi}_{(2)}\gamma^0, \quad (2.101)$$

$$\bar{\lambda}_{(2)} = \bar{\psi}_{(2)}(\overleftarrow{\partial}_a\gamma^a - im_2)\gamma^0 - ig\varphi\bar{\psi}_{(1)}\gamma^0, \quad (2.102)$$

$$\lambda_{(1)} = -\gamma^0(\gamma^a\partial_a + im_1)\psi_{(1)} - ig\varphi\gamma^0\psi_{(2)}, \quad (2.103)$$

$$\lambda_{(2)} = -\gamma^0(\gamma^a\partial_a + im_2)\psi_{(2)} - ig\varphi\gamma^0\psi_{(1)}. \quad (2.104)$$

There are no secondary constraints. Taking suitable linear combinations of constraints, one has to find all numbers of second-class ones, there are

$$\Phi_i = H'_{(i)} = p_{(i)} - i\bar{\psi}_{(i)}\gamma^0, \quad i = 1, 2, \quad (2.105)$$

$$\Phi_3 = \overline{H}'_{(1)} = \bar{p}_{(1)}, \quad (2.106)$$

and

$$\Phi_4 = \overline{H}'_{(2)} = \overline{p}_{(2)}. \quad (2.107)$$

The equations of motion are read as

$$\dot{\varphi} = \{\varphi, H_T\} = p_\varphi, \quad (2.108)$$

$$\dot{\psi}_{(i)} = \{\psi_{(i)}, H_T\} = \lambda_{(i)}, \quad (2.109)$$

$$\dot{\overline{\psi}}_{(i)} = \{\overline{\psi}_{(i)}, H_T\} = \overline{\lambda}_{(i)}, \quad (2.110)$$

$$\dot{p}_\varphi = \{p_\varphi, H_T\} = m^2\varphi + \frac{1}{2}\lambda\varphi^2 + g(\overline{\psi}_{(1)}\psi_{(2)} + \overline{\psi}_{(2)}\psi_{(1)}), \quad (2.111)$$

$$\dot{p}_{(1)} = \{p_{(1)}, H_T\} = \overline{\psi}_{(1)}(i\overleftarrow{\partial}_a\gamma^a + m_1) + g\varphi\overline{\psi}_{(2)}, \quad (2.112)$$

$$\dot{p}_{(2)} = \{p_{(2)}, H_T\} = \overline{\psi}_{(2)}(i\overleftarrow{\partial}_a\gamma^a + m_2) + g\varphi\overline{\psi}_{(1)}, \quad (2.113)$$

$$\dot{\overline{p}}_{(1)} = \{\overline{p}_{(1)}, H_T\} = -(i\gamma^a\partial_a - m_1)\psi_{(1)} + g\varphi\psi_{(2)} - i\gamma^0\lambda_{(1)}, \quad (2.114)$$

$$\dot{\overline{p}}_{(2)} = \{\overline{p}_{(2)}, H_T\} = -(i\gamma^a\partial_a - m_2)\psi_{(2)} + g\varphi\psi_{(1)} - i\gamma^0\lambda_{(2)}. \quad (2.115)$$

Differentiate eq.(2.108) with respect to time, and substituting from eq.(2.111), we get

$$\ddot{\varphi} - m^2\varphi - \frac{1}{2}\lambda\varphi^2 - g(\overline{\psi}_{(1)}\psi_{(2)} + \overline{\psi}_{(2)}\psi_{(1)}) = 0. \quad (2.116)$$

Substituting from eqs.(2.103) and (2.104) into eqs.(2.109), (2.114) and (2.115), we get

$$(i\gamma^\mu\partial_\mu - m_1)\psi_{(1)} - g\varphi\psi_{(2)} = 0, \quad (2.117)$$

$$(i\gamma^\mu \partial_\mu - m_2)\psi_{(2)} - g\varphi \psi_{(1)} = 0, \quad (2.118)$$

$$\dot{\bar{p}}_{(i)} = 0, \quad i = 1, 2. \quad (2.119)$$

Also substituting from eqs.(2.101) and (2.102) into eq.(2.110), we have

$$\partial_0 \bar{\psi}_{(1)} i\gamma^0 - \bar{\psi}_{(1)} (i\overleftarrow{\partial}_a \gamma^a + m_1) - g\varphi \bar{\psi}_{(2)} = 0, \quad (2.120)$$

$$\partial_0 \bar{\psi}_{(2)} i\gamma^0 - \bar{\psi}_{(2)} (i\overleftarrow{\partial}_a \gamma^a + m_2) - g\varphi \bar{\psi}_{(1)} = 0. \quad (2.121)$$

From previous two sections, we obtain the same equations of motion.

2.4 A Non-Abelian Yang-Mills Theories

2.4.1 Non-Abelian Gauge Theories

Quantum Electrodynamics is an example of a U(1) gauge theory. U(1) is the group of the unimodular complex numbers and determines the transformation of the charged fields

$$\Psi(x) \rightarrow \exp(-iq\Lambda(x))\Psi(x) \equiv g(x)\Psi(x). \quad (2.122)$$

It forms a group, which means that for any two elements $g, h \in U(1)$, the product is also in U(1). Furthermore, any element has an inverse g^{-1} , which satisfies $gg^{-1} = g^{-1}g = 1$. The unit 1 satisfies $g1 = 1g = g$, for any $g, h \in U(1)$, $gh = hg$.

It is now tempting to generalize this to other, in general non-commutative groups, which are called non-Abelian groups. It was the way how Yang and Mills discovered SU(2) gauge in 1954. Like for U(1) gauge theories they made the SU(2) transformation into a local one, where at every point the field can be transformed independently. (It should be noted that they were originally after describing the isospin symmetry, that relates protons to neutrons, which form a so-called isospin doublet.)

The simplest non-Abelian gauge group, for which no longer $gh = hg$, is SU(2). This group is well-known from the description of spin one-half particles. It has a two dimensional (spinor) representation, which can also be seen as a representation of the rotation group SO(3). As a local gauge theory it does no longer act on the spinor indices, but on indices related to some internal space, giving rise to so-called internal symmetries. The way the gauge group G acts on fields Ψ is described by a representation of the group G . A representation defines a mapping ρ from G to the space of linear mappings $\text{Map}(V)$, of the linear vector space V into itself.

$$\rho : G \rightarrow \text{Map}(V), \quad \rho(g) : V \rightarrow V. \quad (2.123)$$

Mostly, V will be \mathbb{R}^n , in which case $\rho(g)$ is resp. a real or a complex $n \times n$ matrix. For ρ to be a representation, it has to preserve the group structure of G

$$\rho(g)\rho(h) = \rho(gh), \quad \rho(1) = id_V, \quad (2.124)$$

where id_V means that identical V .

We will generally restrict the gauge symmetries to Lie-groups for which one can write any group element as an exponential of a Lie-algebra element

$$g \equiv \exp(X), \quad X \in L_G. \quad (2.125)$$

This Lie-algebra has a non-commutative, antisymmetric bilinear product

$$(X, Y) \in L_G \times L_G \rightarrow [X, Y] \in L_G. \quad (2.126)$$

2.4.2 Hamilton-Jacobi Formulation of a Non-Abelian Yang-Mills Theories

Consider the Lagrangian density for A non-Abelian theory of fermions interacting with gauge bosons as

$$L = -\frac{1}{4} (F_{\mu\nu}^a)^2 + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi + \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2, \quad (2.127)$$

where ξ can be any finite constant.

In Eq.(2.127) $F_{\mu\nu}^a$ is given by the formula

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c, \quad (2.128)$$

where g represents the coupling constant and f^{abc} are the structure constants of the Lie algebra obey Jacobi identity that is

$$f^{abc} = \varepsilon^{abc},$$

hence ε^{abc} is three dimensional Levi-Civita symbol defined by

$$\varepsilon^{123} = \varepsilon^{231} = \varepsilon^{312} = 1,$$

$$\varepsilon^{132} = \varepsilon^{213} = \varepsilon^{321} = -1,$$

$$\text{all other } \varepsilon^{abc} = 0.$$

The generalized momenta (1.31,1.32) are

$$\pi_a^i = \frac{\vec{\partial} L}{\partial \dot{A}_i^a} = -F_a^{0i}, \quad (2.129)$$

$$\pi_a^0 = \frac{\vec{\partial} L}{\partial \dot{A}_0^a} = \frac{1}{\xi} \partial^\mu A_\mu^a, \quad (2.130)$$

$$p_\psi = \frac{\vec{\partial} L}{\partial \dot{\psi}} = i \bar{\psi} \gamma^0 = -H_\psi, \quad (2.131)$$

$$p_{\bar{\psi}} = \frac{\vec{\partial} L}{\partial \dot{\bar{\psi}}} = 0 = -H_{\bar{\psi}}, \quad (2.132)$$

$$p_\mu = \frac{\vec{\partial} L}{\partial \dot{A}_\mu} = 0 = -H_\mu, \quad (2.133)$$

where we must call attention to necessity of being careful with the spinor indexes. Considering, as usual ψ as a column vector and $\bar{\psi}$ as a row vector implies that p_ψ will be a row vector while $p_{\bar{\psi}}$ will be a column vector.

Equations (2.129) and (2.130), respectively lead us to express the velocities \dot{A}_a^i and \dot{A}_a^0 as

$$\dot{A}_a^i = \pi_i^a - \partial_i A_a^0 + g f_{abc} A_b^0 A_c^i, \quad (2.134)$$

$$\dot{A}_a^0 = \xi \pi_a^0 - \partial_i A_i^a. \quad (2.135)$$

The Hamiltonian density is given by

$$\begin{aligned} H_0 = & \frac{1}{2} \pi_i^a \pi_i^a - \pi_i^a \partial_i A_a^0 - g f^{abc} \pi_a^i A_0^b A_i^c + \frac{1}{2} \xi \pi_0^a \pi_a^0 - \pi_0^a \partial_i A_i^a \\ & + \frac{1}{4} F_a^{ij} F_{ij}^a - \bar{\psi} (i \gamma^i \partial_i + e \gamma^\mu A_\mu - m) \psi. \end{aligned} \quad (2.136)$$

The set of Hamilton-Jacobi Partial Differential Equations (HJPDE) (1.38) read as

$$H'_0 = \pi_4^a + H_0 = 0, \quad (2.137)$$

$$H'_\psi = p_\psi + H_\psi = p_\psi - i \bar{\psi} \gamma^0 = 0, \quad (2.138)$$

$$H'_{\bar{\psi}} = p_{\bar{\psi}} + H_{\bar{\psi}} = p_{\bar{\psi}} = 0, \quad (2.139)$$

$$H'_\mu = p_\mu + H_\mu = p_\mu = 0. \quad (2.140)$$

The equations of motion are obtained as total differential equations as follows:

$$\begin{aligned} dA_a^i &= \frac{\partial H'_0}{\partial \pi_i^a} dt + \frac{\partial H'_\psi}{\partial \pi_i^a} d\psi + \frac{\partial H'_{\bar{\psi}}}{\partial \pi_i^a} d\bar{\psi} + \frac{\partial H'_\mu}{\partial \pi_i^a} dA_\mu, \\ &= [\pi_i^a - \partial_i A_0^a + g f^{abc} A_0^b A_i^c] dt, \end{aligned} \quad (2.141)$$

$$\begin{aligned} dA_a^0 &= \frac{\partial H'_0}{\partial \pi_0^a} dt + \frac{\partial H'_\psi}{\partial \pi_0^a} d\psi + \frac{\partial H'_{\bar{\psi}}}{\partial \pi_0^a} d\bar{\psi} + \frac{\partial H'_\mu}{\partial \pi_0^a} dA_\mu, \\ &= [\xi \pi_0^a - \partial^i A_i^a] dt, \end{aligned} \quad (2.142)$$

$$\begin{aligned} d\pi_a^i &= -\frac{\partial H'_0}{\partial A_i^a} dt - \frac{\partial H'_\psi}{\partial A_i^a} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial A_i^a} d\bar{\psi} - \frac{\partial H'_\mu}{\partial A_i^a} dA_\mu, \\ &= [g f^{abc} \pi_c^i A_0^b - \partial_l (F_a^{li} + \pi_0^a) - F_a^{il} g f^{abc} A_c^b] dt, \end{aligned} \quad (2.143)$$

$$\begin{aligned} d\pi_a^0 &= -\frac{\partial H'_0}{\partial A_0^a} dt - \frac{\partial H'_\psi}{\partial A_0^a} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial A_0^a} d\bar{\psi} - \frac{\partial H'_\mu}{\partial A_0^a} dA_\mu, \\ &= [\partial_i \pi_a^i + g f^{abc} \pi_b^i A_i^c] dt, \end{aligned} \quad (2.144)$$

$$\begin{aligned} dp_\psi &= -\frac{\partial H'_0}{\partial \psi} dt - \frac{\partial H'_\psi}{\partial \psi} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial \psi} d\bar{\psi} - \frac{\partial H'_\mu}{\partial \psi} dA_\mu, \\ &= [-\bar{\psi} (i \overleftarrow{\partial}_i \gamma^i - e \gamma^\mu A_\mu + m)] dt, \end{aligned} \quad (2.145)$$

$$\begin{aligned}
dp_{\bar{\psi}} &= -\frac{\partial H'_0}{\partial \bar{\psi}} dt - \frac{\partial H'_\psi}{\partial \bar{\psi}} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial \bar{\psi}} d\bar{\psi} - \frac{\partial H'_\mu}{\partial \bar{\psi}} dA_\mu, \\
&= [(i\gamma^i \partial_i + e\gamma^\mu A_\mu - m)\psi] dt + i\gamma^0 d\psi, \tag{2.146}
\end{aligned}$$

$$\begin{aligned}
dp_\mu &= -\frac{\partial H'_0}{\partial A_\mu} dt - \frac{\partial H'_\psi}{\partial A_\mu} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial A_\mu} d\bar{\psi} - \frac{\partial H'_\mu}{\partial A_\mu} dA_\mu, \\
&= (\bar{\psi} e \gamma^\mu \psi) dt, \tag{2.147}
\end{aligned}$$

$$d\pi_4^a = -\frac{\partial H'_0}{\partial t} dt - \frac{\partial H'_\psi}{\partial t} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial t} d\bar{\psi} - \frac{\partial H'_\mu}{\partial t} dA_\mu. \tag{2.148}$$

The integrability conditions imply that the variation of the constraints H'_ψ , $H'_{\bar{\psi}}$ and H'_μ should be identically zero; that is

$$dH'_\psi = dp_\psi - i d\bar{\psi} \gamma^0 = 0, \tag{2.149}$$

$$dH'_{\bar{\psi}} = dp_{\bar{\psi}} = 0, \tag{2.150}$$

$$dH'_\mu = dp_\mu = 0. \tag{2.151}$$

The vanishing of total differential of H'_μ leads to a new constraint

$$H''_\mu = \bar{\psi} e \gamma^\mu \psi. \tag{2.152}$$

When we taking a gain the total differential of H''_μ , we notice it vanishes identically

$$dH''_\mu = 0. \tag{2.153}$$

From Eqs.(2.141) and (2.142), respectively we obtain

$$\dot{A}_a^i = \pi_a^i - \partial_i A_0^a + g f^{abc} A_0^b A_i^c, \quad (2.154)$$

$$\dot{A}_0^i = \xi \pi_0^a - \partial^i A_i^a. \quad (2.155)$$

Substituting from Eqs. (2.145) and (2.146) into Eqs. (2.149) and (2.150), respectively, we get

$$\bar{\psi} (i \overleftarrow{\partial}_\mu \gamma^\mu - e \gamma^\mu A_\mu + m) = 0, \quad (2.156)$$

$$(i \gamma^\mu \partial_\mu + e \gamma^\mu A_\mu - m) \psi = 0. \quad (2.157)$$

Also from Eqs. (2.143 - 2.145, 2.147), we get the following equations of motion:

$$\dot{\pi}_a^i = g f^{abc} \pi_c^i A_0^b - \partial_l (F_a^{li} + \pi_0^a) - F_a^{il} g f^{abc} A_c^b, \quad (2.158)$$

$$\dot{\pi}_a^0 = \partial_i \pi_a^i + g f^{abc} \pi_b^i A_i^c, \quad (2.159)$$

$$\dot{p}_\psi = -\bar{\psi} (i \overleftarrow{\partial}_i \gamma^i - e \gamma^\mu A_\mu + m), \quad (2.160)$$

$$\dot{p}_\mu = \bar{\psi} e \gamma^\mu \psi. \quad (2.161)$$

Substituting from Eq. (2.157) into Eq.(2.146), we have

$$\dot{p}_{\bar{\psi}} = 0. \quad (2.162)$$

In the following section the same system will be discussed using Dirac's approach.

2.4.3 Dirac's Formulation of a Non-Abelian Yang-Mills Theories

The total Hamiltonian is given as

$$\begin{aligned}
H_T = & \frac{1}{2} \pi_i^a \pi_i^a - \pi_i^a \partial_i A_0^a - g f^{abc} \pi_a^i A_0^b A_i^c + \frac{1}{2} \xi \pi_0^a \pi_a^0 - \pi_0^a \partial_i A_i^a \\
& + \frac{1}{4} F_a^{ij} F_{ij}^a - \bar{\psi} (i \gamma^i \partial_i + e \gamma^\mu A_\mu - m) \psi \\
& + \lambda_\psi (p_\psi - i \gamma^0 \bar{\psi}) + \lambda_{\bar{\psi}} p_{\bar{\psi}} + \lambda_\mu p_\mu, \quad (2.163)
\end{aligned}$$

where λ_ψ , $\lambda_{\bar{\psi}}$ and λ_μ are Lagrange multipliers to be determined. From the consistency conditions, the time derivative of the primary constraints should be zero, that is

$$\dot{H}'_\psi = \{H'_\psi, H_T\} = -\bar{\psi} (i \overleftarrow{\partial}_i \gamma^i - e \gamma^\mu A_\mu + m) - i \lambda_{\bar{\psi}} \gamma^0 \approx 0, \quad (2.164)$$

$$\dot{H}'_{\bar{\psi}} = \{H'_{\bar{\psi}}, H_T\} = (i \gamma^i \partial_i + e \gamma^\mu A_\mu - m) \psi + i \gamma^0 \lambda_\psi \approx 0, \quad (2.165)$$

$$\dot{H}'_\mu = \{H'_\mu, H_T\} = \bar{\psi} e \gamma^\mu \psi \approx 0. \quad (2.166)$$

Relations (2.164) and (2.165) fix the multipliers $\lambda_{\bar{\psi}}$ and λ_ψ respectively as

$$\lambda_{\bar{\psi}} = i \bar{\psi} (i \overleftarrow{\partial}_i \gamma^i - e \gamma^\mu A_\mu + m) \gamma^0, \quad (2.167)$$

$$\lambda_\psi = i \gamma^0 (i \gamma^i \partial_i + e \gamma^\mu A_\mu - m) \psi. \quad (2.168)$$

Eq.(2.166) lead to the secondary constraint

$$H''_\mu = \bar{\psi} e \gamma^\mu \psi \approx 0. \quad (2.169)$$

There are no tertiary constraints, since

$$\dot{H}''_{\mu} = \{H''_{\mu}, H_T\} = 0. \quad (2.170)$$

By taking suitable linear combinations of constraints, one has to find the first-class, that is

$$\Phi_1 = H'_{\mu} = p_{\mu}, \quad (2.171)$$

whereas the constraints

$$\Phi_2 = H'_{\psi} = p_{\psi} - i \gamma^0 \bar{\psi}, \quad (2.172)$$

$$\Phi_3 = H'_{\bar{\psi}} = p_{\bar{\psi}}, \quad (2.173)$$

$$\Phi_4 = H''_{\mu} = \bar{\psi} e \gamma^{\mu} \psi = 0, \quad (2.174)$$

are second-class.

In Dirac's method, Poisson brackets lead us to the following equations of motion:

$$\dot{A}_0^a = \{A_0^a, H_T\} = \xi \pi_0^a - \partial^i A_i^a, \quad (2.175)$$

$$\dot{A}_i^a = \{A_i^a, H_T\} = \pi_i^a - \partial_i A_0^a + g f^{abc} A_0^b A_i^c, \quad (2.176)$$

$$\dot{\psi} = \{\psi, H_T\} = \lambda_{\psi}, \quad (2.177)$$

$$\dot{\bar{\psi}} = \{\bar{\psi}, H_T\} = \lambda_{\bar{\psi}}, \quad (2.178)$$

$$\dot{A}_{\mu} = \{A_{\mu}, H_T\} = \lambda_{\mu}, \quad (2.179)$$

$$\dot{\pi}_a^0 = \{\pi_a^0, H_T\} = \partial_i \pi_a^i + g f^{abc} \pi_b^i A_i^c, \quad (2.180)$$

$$\dot{\pi}_a^i = \{\pi_a^i, H_T\} = g f^{abc} \pi_c^i A_0^b - \partial_l (F_a^{li} + \pi_0^a) - F_a^{il} g f^{abc} A_c^b, \quad (2.181)$$

$$\dot{p}_\psi = \{p_\psi, H_T\} = -\bar{\psi} (i \overleftarrow{\partial}_i \gamma^i - e \gamma^\mu A_\mu + m), \quad (2.182)$$

$$\dot{p}_{\bar{\psi}} = \{p_{\bar{\psi}}, H_T\} = (i \gamma^i \partial_i + e \gamma^\mu A_\mu - m) \psi + i \gamma^0 \lambda_\psi, \quad (2.183)$$

$$\dot{p}_\mu = \{p_\mu, H_T\} = \bar{\psi} e \gamma^\mu \psi. \quad (2.184)$$

Substituting from Eq. (2.168) into Eqs. (2.177) and (2.183), we get

$$(i \gamma^\mu \partial_\mu + e \gamma^\mu A_\mu - m) \psi = 0, \quad (2.185)$$

$$\dot{p}_{\bar{\psi}} = 0, \quad (2.186)$$

and from Eq.(2.167) into (2.178), we have

$$\bar{\psi} (i \overleftarrow{\partial}_\mu \gamma^\mu - e \gamma^\mu A_\mu + m) = 0. \quad (2.187)$$

We will contact ourselves with a partial gauge fixing by introducing gauge constraints for the first-class primary constraints only, just to fix the multiplier λ_μ in Eq.(2.163). Since p_μ is vanishing weakly, a gauge choice near at hand would be

$$\phi_1' = A_\mu = 0. \quad (2.188)$$

But for this forbids dynamics at all, since the requirement $\dot{A}_\mu = 0$ implies $\lambda_\mu = 0$.

As a comparison between the last two sections, we get the fact that Hamilton-Jacobi method and Dirac's method gives the same equations of motion.

Chapter 3

Path Integral Quantization

In this chapter we shall give a brief review of the Faddeev's, Senjanovic's and Hamilton-Jacobi quantization to give the path integral quantization of constrained system.

3.1 Faddeev Popov Method

The classical dynamics of an n -dimensional system is determined by the Lagrangian, a function of the n coordinates and their time derivatives. From the Lagrangian, we can construct the Hamiltonian, which is a function of the phase space. In canonical quantization, the Hamiltonian becomes an operator which acts on Hilbert space which is built from the n coordinates. The Hamiltonian is a generator of time translations and thus determine quantum dynamics.

For a system with n degrees of freedom and having α first-class constraints ϕ_a , but no second-class constraints, Faddeev has formulated the transition amplitude as [6]

$$\langle Out | S | In \rangle = \int \exp \left[i \int_{-\infty}^{\infty} (p_i \dot{q}_i - H_0) dt \right] \prod_t d\mu(q_i(t), p_i(t)), \quad (3.1)$$

where H_0 is the Hamiltonian of the system. The measure of integration is defined by

$$d\mu(q_i(t), p_i(t)) = \left(\prod_{a=1}^{\alpha} \delta(\chi_a) \delta(\phi_a) \right) \det ||\{\chi_a, \phi_a\}|| \prod_{i=1}^n dp_i dq^i. \quad (3.2)$$

and $\chi_a(p_i, q_i)$ are the gauge-fixing condition with

1. $\{\chi_a, \chi_{a'}\} = 0$,
2. $\det ||\{\chi_a, \phi_a\}|| \neq 0$.

3.2 Senjanovic Method

In this section we shall generalize Faddeev's method to the case when second-class constraints are present. This generalization is called Senjanovic method.

Consider a mechanical system with α first-class constraints ϕ_a , β second-class constraints θ_b , and the gauge conditions associated with the first-class constraints χ_a . Let the χ_a be chosen

in such a way that $\{\chi_a, \chi_b\} = 0$.

Then the expression for the S -matrix element is [8]

$$\langle Out | S | In \rangle = \int \exp \left[i \int_{-\infty}^{\infty} (p_i \dot{q}_i - H_0) dt \right] \prod_t d\mu(q(t), p(t)), \quad (3.3)$$

and

$$d\mu(q, p) = \left(\prod_{a=1}^{\alpha} \delta(\chi_a) \delta(\phi_a) \right) det ||\{\chi_a, \phi_a\}|| \times \prod_{b=1}^{\beta} \delta(\theta_b) det ||\{\theta_a, \theta_b\}||^{\frac{1}{2}} \prod_{i=1}^n dp_i dq^i. \quad (3.4)$$

where H_0 is the Hamiltonian of the system and $d\mu(q, p)$ is the measure of integration.

3.3 Hamilton-Jacobi Quantization

In this section, we shall study the path integral formulation of the constrained systems given in refs. [22-26].

Let us consider a singular Lagrangian $L = L(q_i, \dot{q}_i, \tau)$, $i = 1, \dots, n$, with the Hess matrix defined in (1.16) of rank $(n - r)$, $r < n$. The generalized momenta p_i corresponding to the generalized coordinates q_i are defined in (1.31) and (1.32). Since the rank of Hess matrix is $(n - r)$, one may solve (1.31) for \dot{q}_a defined in (1.33). The canonical Hamiltonian H_0 defined in (1.35),

and the set of HJPDE is expressed in (1.36-1.37). As we define $p_\beta = \frac{\partial S[q_a; x_\alpha]}{\partial x_\beta}$ and $p_a = \frac{\partial S[q_a; x_\alpha]}{\partial q_a}$ with $x_0 = t$ and S being the action. The total differential equations given in (1.41-1.44) are integrable if (1.45) and (1.46) are hold [22]. If conditions (1.45) and (1.46) are not satisfied identically, one considers them as new constraints and again consider their variations.

Thus, repeating this procedure one may obtain a set of constraints such that all variations vanish. Simultaneous solutions of canonical equations with all these constraints provide to obtain the set of canonical phase space coordinates (q_a, p_a) as functions of t_α , besides the canonical action integral is obtained in terms of the canonical coordinates. H'_α can be interpreted as the infinitesimal generator of canonical transformations given by parameters t_α . In this case path integral representation may be written as [22-26]

$$\langle Out | S | In \rangle = \int \prod_{a=1}^{n-p} dq^a dp^a \exp \left\{ \int_{t_\alpha}^{t'_\alpha} \left(-H_\alpha + p_\alpha \frac{\partial H'_\alpha}{\partial p_\alpha} \right) dt_\alpha \right\},$$

$$a = 1, \dots, n-p, \quad \alpha = 0, n-p+1, \dots, n. \quad (3.5)$$

In fact, this path integral is an integration over the canonical phase-space coordinates (q^a, p^a) .

Chapter 4

Applications on Path integral Quantization of Fields

In this chapter we will study the Hamilton-Jacobi quantization of the actual physical systems, which illustrate the basic concepts of the proceeding chapter.

In Section **4.1** we will consider one application on both Faddeev and Hamilton-Jacobi quantization, in Sections **4.2** we will consider one applications on both Senjanovic and Hamilton-Jacobi quantization, and in section **4.3** we will consider an application on Hamilton-Jacobi quantization.

4.1 Quantization of The Scalar Field Coupled Minimally to The Vector Potential

Consider the action integral for the scalar field coupled minimally to the vector potential as

$$S = \int d_4x L, \quad (4.1)$$

where the Lagrangian L is given by

$$L = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + (D_\mu \varphi)^*(x) D^\mu \varphi(x) - m^2 \varphi^*(x) \varphi(x), \quad (4.2)$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (4.3)$$

and

$$D_\mu \varphi(x) = \partial_\mu \varphi(x) - ie A_\mu(x) \varphi(x). \quad (4.4)$$

Let us first use Hamilton-Jacobi path integral quantization.

The canonical momenta are obtained as

$$\pi^i = \frac{\overrightarrow{\partial} L}{\partial \dot{A}_i} = -F^{0i}, \quad (4.5)$$

$$\pi^0 = \frac{\overrightarrow{\partial} L}{\partial \dot{A}_0} = 0, \quad (4.6)$$

$$p_\varphi = \frac{\overrightarrow{\partial} L}{\partial \dot{\varphi}} = (D_0 \varphi)^* = \dot{\varphi}^* + ie A_0 \varphi^*, \quad (4.7)$$

$$p_{\varphi^*} = \frac{\overrightarrow{\partial} L}{\partial \dot{\varphi}^*} = (D_0 \varphi) = \dot{\varphi} - i e A_0 \varphi, \quad (4.8)$$

From Eqs. (4.5), (4.7) and (4.8), the velocities $\dot{A}_i, \dot{\varphi}^*$ and $\dot{\varphi}$ can be expressed in terms of momenta π_i, p_φ and p_{φ^*} respectively as

$$\dot{A}_i = -\pi_i - \partial_i A_0, \quad (4.9)$$

$$\dot{\varphi}^* = p_\varphi - i e A_0 \varphi^*, \quad (4.10)$$

$$\dot{\varphi} = p_{\varphi^*} + i e A_0 \varphi. \quad (4.11)$$

The canonical Hamiltonian H_0 is obtained as

$$\begin{aligned} H_0 = & \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi_i \pi^i + \pi^i \partial_i A_0 + p_{\varphi^*} p_\varphi + i e A_0 \varphi p_\varphi \\ & - i e A_0 \varphi^* p_{\varphi^*} - (D_i \varphi)^* (D^i \varphi) + m^2 \varphi^* \varphi. \end{aligned} \quad (4.12)$$

The set of Hamilton-Jacobi Partial Differential Equations (HJPDE) (1.38) read as

$$H'_0 = \pi_4 + H_0, \quad (4.13)$$

$$H' = \pi_0 + H = \pi_0 = 0. \quad (4.14)$$

Therefore, the total differential equations for the characteristic (1.41-1.43) are written in the following forms:

$$dA^i = \frac{\partial H'_0}{\partial \pi_i} dt + \frac{\partial H'}{\partial \pi_i} dA^0,$$

$$= -(\pi^i + \partial_i A_0) dt, \quad (4.15)$$

$$dA^0 = \frac{\partial H'_0}{\partial \pi_0} dt + \frac{\partial H'}{\partial \pi_0} dA^0 = dA^0, \quad (4.16)$$

$$\begin{aligned} d\varphi &= \frac{\partial H'_0}{\partial p_\varphi} dt + \frac{\partial H'}{\partial p_\varphi} dA^0, \\ &= (p_{\varphi^*} + ieA_0\varphi) dt, \end{aligned} \quad (4.17)$$

$$\begin{aligned} d\varphi^* &= \frac{\partial H'_0}{\partial p_{\varphi^*}} dt + \frac{\partial H'}{\partial p_{\varphi^*}} dA^0, \\ &= (p_\varphi - ieA_0\varphi^*) dt, \end{aligned} \quad (4.18)$$

$$\begin{aligned} d\pi^i &= -\frac{\partial H'_0}{\partial A_i} dt - \frac{\partial H'}{\partial A_i} dA^0, \\ &= [\partial_l F^{li} + ie(\varphi^* \partial^i \varphi + \varphi \partial_i \varphi^*) + 2e^2 A^i \varphi \varphi^*] dt, \end{aligned} \quad (4.19)$$

$$\begin{aligned} d\pi^0 &= -\frac{\partial H'_0}{\partial A_0} dt - \frac{\partial H'}{\partial A_0} dA^0, \\ &= [\partial_i \pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi p_\varphi] dt, \end{aligned} \quad (4.20)$$

$$\begin{aligned} dp_\varphi &= -\frac{\partial H'_0}{\partial \varphi} dt - \frac{\partial H'}{\partial \varphi} dA^0, \\ &= [(\vec{D} \cdot \vec{D} \varphi)^* - m^2 \varphi^* - ieA_0 p_\varphi] dt, \end{aligned} \quad (4.21)$$

and

$$dp_{\varphi^*} = -\frac{\partial H'_0}{\partial \varphi^*} dt - \frac{\partial H'}{\partial \varphi^*} dA^0,$$

$$= [(\vec{D} \cdot \vec{D}\varphi) - m^2\varphi + ieA_0p_{\varphi^*}] dt. \quad (4.22)$$

The integrability condition ($dH'_\alpha = 0$) implies that the variation of the constraint H' should be identically zero, that is

$$dH' = d\pi_0 = 0, \quad (4.23)$$

which leads to a new constraint

$$H'' = \partial_i \pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi p_\varphi = 0. \quad (4.24)$$

Taking the total differential of H'' , we have

$$dH'' = \partial_i d\pi^i + iep_{\varphi^*} d\varphi^* + ie\varphi^* dp_{\varphi^*} - ie\varphi dp_\varphi - iep_\varphi d\varphi = 0. \quad (4.25)$$

Then the set of equations (4.15 - 4.22) is integrable. Therefore, the canonical phase space coordinates (φ, p_φ) and $(\varphi^*, p_{\varphi^*})$ are obtained in terms of parameters (t, A^0) .

Making use of Eq.(1.44) and (4.12 - 4.14), we obtain the canonical action integral as

$$Z = \int d^4x \left(-\frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi_i \pi^i + p_\varphi p_{\varphi^*} + \vec{D}\varphi^* \cdot \vec{D}\varphi + m^2 |\varphi|^2 \right), \quad (4.26)$$

where

$$\vec{D} = \vec{\nabla} + ie\vec{A}. \quad (4.27)$$

Now the path integral representation (3.5) is given by

$$\begin{aligned} \langle out|S|In\rangle = \int \prod_i dA^i d\pi^i d\varphi dp_\varphi d\varphi^* dp_{\varphi^*} \exp \left[i \left\{ \int d^4x \right. \right. \\ \left. \left. \left(-\frac{1}{2} \pi_i \pi^i - \frac{1}{4} F^{ij} F_{ij} + p_\varphi p_{\varphi^*} + (D_i \varphi)^* (D_i \varphi) - m^2 \varphi^* \varphi \right) \right\} \right]. \end{aligned} \quad (4.28)$$

Secondly, we apply the Faddeev method to the pervious model, we start with the total Hamiltonian

$$\begin{aligned} H_T = \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi_i \pi^i + \pi^i \partial_i A_0 + p_{\varphi^*} p_\varphi + ie A_0 \varphi p_\varphi \\ - ie A_0 \varphi^* p_{\varphi^*} - (D_i \varphi)^* (D^i \varphi) + m^2 \varphi^* \varphi + \lambda \pi_0. \end{aligned} \quad (4.29)$$

According to Dirac's method, the time derivative of the primary constraints should be zero, that is

$$\dot{H}' = \{H', H_T\} = \partial_i \pi^i + ie \varphi^* p_{\varphi^*} - ie \varphi p_\varphi \approx 0, \quad (4.30)$$

which leads to the secondary constraints

$$H'' = \partial_i \pi^i + ie \varphi^* p_{\varphi^*} - ie \varphi p_\varphi \approx 0. \quad (4.31)$$

There are no tertiary constraints, since

$$\dot{H}'' = \{H'', H_T\} = 0. \quad (4.32)$$

By taking suitable linear combinations of constraints, one has to find the first-class one, that is

$$\Phi = H' = \pi_0. \quad (4.33)$$

The equations of motion read as

$$\dot{A}^i = \{A_0^i, H_T\} = -(\pi^i + \partial_i A_0), \quad (4.34)$$

$$\dot{A}^0 = \{A^0, H_T\} = \lambda, \quad (4.35)$$

$$\dot{\varphi} = \{\varphi, H_T\} = (p_{\varphi^*} + ieA_0\varphi), \quad (4.36)$$

$$\dot{\varphi}^* = \{\varphi^*, H_T\} = (p_{\varphi} - ieA_0\varphi^*), \quad (4.37)$$

$$\dot{\pi}^i = \{\pi^i, H_T\} = \partial_l F^{li} + ie(\varphi^* \partial^i \varphi + \varphi \partial_i \varphi^*) + 2e^2 A^i \varphi \varphi^*, \quad (4.38)$$

$$\dot{\pi}^0 = \{\pi^0, H_T\} = \partial_i \pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi p_{\varphi}, \quad (4.39)$$

$$\dot{p}_{\varphi} = \{p_{\varphi}, H_T\} = (\vec{D} \cdot \vec{D} \varphi)^* - m^2 \varphi^* - ieA_0 p_{\varphi}, \quad (4.40)$$

$$\dot{p}_{\varphi^*} = \{p_{\varphi^*}, H_T\} = (\vec{D} \cdot \vec{D} \varphi) - m^2 \varphi + ieA_0 p_{\varphi^*}. \quad (4.41)$$

We will contact ourselves with a partial gauge fixing by introducing gauge constraints for the first-class primary constraints

only, just to fix the multiplier λ in Eq.(4.29). Since there are weakly vanishing a gauge choice near at hand would be:

$$\phi' = A_0 = 0. \quad (4.42)$$

Making use of Eq.(3.1), we obtain the path integral quantization

$$\begin{aligned} \langle out|S|In\rangle = \int \exp \left[i \int_{-\infty}^{+\infty} \left(-\frac{1}{2} \pi_i \pi^i - \frac{1}{4} F^{ij} F_{ij} + p_\varphi p_{\varphi^*} \right. \right. \\ \left. \left. + \vec{D}\varphi^* \cdot \vec{D}\varphi - m^2 |\varphi|^2 \right) d^4x dA^i d\pi^i d\varphi dp_\varphi d\varphi^* dp_{\varphi^*} \right]. \end{aligned} \quad (4.43)$$

We showed that Eq.(4.28) and Eq.(4.43) are identical.

4.2 The Relativistic Local Free Field Theory

As a second physical example of a singular system described by a first order action, namely a system whose Lagrange function is linear in the velocities. However, the associated constraints are all second-class. Let us consider the relativistic local free field theory of spin $\frac{1}{2}$ in a Minkowski spacetime of dimension D . As usual, spacetime coordinates are denoted as x^μ, y^μ ($\mu = 0, 1, \dots, D-1$) and space components are labelled by $i, j = 1, 2, \dots, D-1$. The Minkowski matrix $\eta^{\mu\nu}$ is chosen with a signature with mostly minus signs, and we also set $\hbar = c = 1$.

The system is described by the first order action

$$S[\psi] = \int d^D x \, l(\psi, \partial_\mu \psi). \quad (4.44)$$

with the local Lagrangian density function

$$l(\psi, \partial_\mu \psi) = i \frac{\lambda + 1}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi + i \frac{\lambda - 1}{2} \partial_\mu \bar{\psi} \gamma^\mu \psi - m \bar{\psi} \psi \quad (4.45)$$

Here λ is a parameter, the matrices γ^μ define the Dirac algebra in D-dimensional Minkowski space-time

$$\{\gamma^\mu, \gamma^\nu\} = 2 \eta^{\mu\nu}, \quad \gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0 \quad (4.46)$$

and $\psi_\alpha(x) (\alpha = 1, 2, \dots, 2^{[D/2]})$ are Grassmann even degrees of freedom defining a Dirac spinor, with

$$\bar{\psi}(x) = \psi^\dagger(x) \gamma^0 \quad (4.47)$$

For simplicity, the fields $\psi(x)$ are assumed to fall off sufficiently rapidly at infinity for all practical purposes.

4.2.1 Quantization of The Relativistic Local Free Field Theory

The Lagrangian (4.45) is singular, since the rank of the Hess matrix (1.16) is zero. The generalized momenta are written as

$$p = \frac{\vec{\partial} L}{\partial \partial_0 \psi} = i \frac{\lambda + 1}{2} \bar{\psi} \gamma^0 = -H, \quad (4.48)$$

and

$$\bar{P} = \frac{\vec{\partial} L}{\partial \partial_0 \bar{\psi}} = i \frac{\lambda - 1}{2} \gamma^0 \psi = -\bar{H}, \quad (4.49)$$

where we must call attention to the necessity of being careful with the spinor indexes. Considering, as usual ψ as a column vector and $\bar{\psi}$ as a row vector implies that p will be a row vector while \bar{p} will be a column vector.

The usual Hamiltonian H_0 is given as

$$H_0 = -L + \partial_0 \psi p + \partial_0 \bar{\psi} \bar{p} \Big|_{p=-H, \bar{p}=-\bar{H}}, \quad (4.50)$$

or,

$$H_0 = -i \frac{\lambda + 1}{2} \bar{\psi} \gamma^a \partial_a \psi - i \frac{\lambda - 1}{2} \partial_a \bar{\psi} \gamma^a \psi + m \bar{\psi} \psi, \quad (4.51)$$

$$a = 1, 2, 3.$$

The set of Hamilton-Jacobi partial differential equations (HJPDE) is

$$H'_0 = p_0 + H_0 = p_0 - i \frac{\lambda + 1}{2} \bar{\psi} \gamma^a \partial_a \psi - i \frac{\lambda - 1}{2} \partial_a \bar{\psi} \gamma^a \psi + m \bar{\psi} \psi, \quad (4.52)$$

$$H' = p + H = p - i \frac{\lambda + 1}{2} \bar{\psi} \gamma^0 = 0, \quad (4.53)$$

$$\bar{H}' = \bar{p} + \bar{H} = \bar{p} - i \frac{\lambda - 1}{2} \gamma^0 \psi = 0. \quad (4.54)$$

Therefor, the total differential equations for the characteristic (1.41 - 1.43) are:

$$\begin{aligned}
d\psi &= \frac{\partial H'_0}{\partial p} dt + \frac{\partial H'}{\partial p} d\psi + \frac{\partial \bar{H}'}{\partial p} d\bar{\psi}, \\
&= d\psi,
\end{aligned} \tag{4.55}$$

$$\begin{aligned}
d\bar{\psi} &= \frac{\partial H'_0}{\partial \bar{p}} dt + \frac{\partial H'}{\partial \bar{p}} d\psi + \frac{\partial \bar{H}'}{\partial \bar{p}} d\bar{\psi}, \\
&= d\bar{\psi},
\end{aligned} \tag{4.56}$$

$$\begin{aligned}
dp &= -\frac{\partial H'_0}{\partial \psi} dt - \frac{\partial H'}{\partial \psi} d\psi - \frac{\partial \bar{H}'}{\partial \psi} d\bar{\psi}, \\
&= (-i\partial_a \bar{\psi} \gamma^a - m\bar{\psi}) d\tau + i\frac{\lambda-1}{2} d\bar{\psi} \gamma^0, \tag{4.57}
\end{aligned}$$

$$\begin{aligned}
d\bar{p} &= -\frac{\partial H'_0}{\partial \bar{\psi}} dt - \frac{\partial H'}{\partial \bar{\psi}} d\psi - \frac{\partial \bar{H}'}{\partial \bar{\psi}} d\bar{\psi}, \\
&= (i\gamma^a \partial_a \psi - m\psi) d\tau + i\frac{\lambda+1}{2} \gamma^0 d\psi. \tag{4.58}
\end{aligned}$$

To check whether the set of equations (4.55 - 4.58) is integrable or not, we have to consider the total variation of the constraints.

In fact

$$dH' = dp - i\frac{\lambda+1}{2} d\bar{\psi} \gamma^0 = 0, \tag{4.59}$$

$$d\bar{H}' = d\bar{p} - i\frac{\lambda-1}{2} \gamma^0 d\psi = 0. \tag{4.60}$$

The constraints (4.53 , 4.54), lead us to obtain

$$d\bar{\psi} = i(i\partial_a \bar{\psi} \gamma^a + m\bar{\psi}) \gamma^0 dt, \tag{4.61}$$

and

$$d\psi = i\gamma^0(i\gamma^a\partial_a\psi - m\psi)dt. \quad (4.62)$$

Then, we conclude that the set of equations (4.55 - 4.58) is integrable.

Making use of (1.44) and (4.52 - 4.54), we can write the canonical action integral as

$$Z = \int d^4x \left(i\frac{\lambda+1}{2} \bar{\psi}\gamma^\mu\partial_\mu\psi + i\frac{\lambda-1}{2} \partial_\mu\bar{\psi}\gamma^\mu\psi - m\bar{\psi}\psi \right). \quad (4.63)$$

Now we turn to the problem of the path integral quantization, where the S-matrix element is given by

$$\begin{aligned} \langle \psi, \bar{\psi}, \iota; \psi', \bar{\psi}', \iota \rangle = \int d\psi d\bar{\psi} \exp \left[i \left\{ \int d^4x \left(i\frac{\lambda+1}{2} \bar{\psi}\gamma^\mu\partial_\mu\psi \right. \right. \right. \\ \left. \left. \left. + i\frac{\lambda-1}{2} \partial_\mu\bar{\psi}\gamma^\mu\psi - m\bar{\psi}\psi \right) \right\} \right]. \quad (4.64) \end{aligned}$$

To check the results obtained using the Hamilton-Jacobi approach, we will study the problem using Dirac's method.

The total Hamiltonian is given as

$$H_T = H_0 + \nu H' + \bar{\nu} \bar{H}', \quad (4.65)$$

or

$$\begin{aligned} H_T = -i\frac{\lambda+1}{2} \bar{\psi}\gamma^a\partial_a\psi - i\frac{\lambda-1}{2} \partial_a\bar{\psi}\gamma^a\psi + m\bar{\psi}\psi \\ + \nu(p - i\frac{\lambda+1}{2} \bar{\psi}\gamma^0) + \bar{\nu}(\bar{p} - i\frac{\lambda-1}{2} \gamma^0\psi), \quad (4.66) \end{aligned}$$

where ν and $\bar{\nu}$ are Lagrange multipliers to be determined. From the consistency conditions, the time derivative of the primary constraints should be zero, that is

$$\dot{H}' = \{H', H_T\} = -i\partial_a \bar{\psi} \gamma^a - m\bar{\psi} - i\bar{\nu} \gamma^0 \approx 0, \quad (4.67)$$

$$\dot{\bar{H}}' = \{\bar{H}', H_T\} = i\gamma^a \partial_a \psi - m\psi + i\gamma^0 \nu \approx 0. \quad (4.68)$$

Eqs. (4.67), and (4.68) fix the multipliers $\bar{\nu}$ and ν , respectively as

$$\bar{\nu} = -\bar{\psi}(\overleftarrow{\partial}_a \gamma^a - im)\gamma^0, \quad (4.69)$$

and

$$\nu = -\gamma^0(\gamma^a \partial_a + im)\psi. \quad (4.70)$$

There are no secondary constraints. By taking suitable linear combinations of constraints, one has to find the maximal number of second class only, there are

$$\Phi_1 = H' = p - i\frac{\lambda+1}{2} \bar{\psi} \gamma^0, \quad (4.71)$$

and

$$\Phi_2 = \bar{H}' = \bar{p} - i\frac{\lambda-1}{2} \gamma^0 \psi. \quad (4.72)$$

The total Hamiltonian is vanishing weakly. It can completely be written in terms of second-class constraints as

$$H_T = -i\frac{\lambda+1}{2} \bar{\psi}\gamma^a\partial_a\psi - i\frac{\lambda-1}{2} \partial_a\bar{\psi}\gamma^a\psi + m\bar{\psi}\psi + \nu\Phi_1 + \bar{\nu}\Phi_2. \quad (4.73)$$

The equations of motion are read as

$$\dot{\psi} = \{\psi, H_T\} = \nu, \quad (4.74)$$

$$\dot{\bar{\psi}} = \{\bar{\psi}, H_T\} = \bar{\nu}, \quad (4.75)$$

$$\dot{p} = \{p, H_T\} = -i\partial_a\bar{\psi}\gamma^a - m\bar{\psi} + i\bar{\nu}\frac{\lambda-1}{2}\gamma^0, \quad (4.76)$$

and

$$\dot{\bar{p}} = \{\bar{p}, H_T\} = i\gamma^a\partial_a\psi - m\psi + i\nu\frac{\lambda+1}{2}\gamma^0. \quad (4.77)$$

To obtain the path integral quantization, taking into our consideration that we have two constraints (primary constraint), which are second-class constraints, then we make use the Senjanovic method Eq.(3.3) one obtains

$$\begin{aligned} \langle Out|S|In\rangle &= \int \exp\left[i \int_{-\infty}^{+\infty} \left(i\frac{\lambda+1}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi + i\frac{\lambda-1}{2}\partial_\mu\bar{\psi}\gamma^\mu\psi - m\bar{\psi}\psi \right) dt D\psi Dp D\bar{\psi} D\bar{p} \det(\gamma^0 I) \right. \\ &\quad \left. \times \delta(p - i\frac{\lambda+1}{2}\bar{\psi}\gamma^0) \delta(\bar{p} - i\frac{\lambda-1}{2}\gamma^0\psi) \right]. \end{aligned} \quad (4.78)$$

After integrating over p and \bar{p} one can arrive at the result (4.64).

4.3 The Electromagnetic Field Coupled to A Spinor

4.3.1 The spinor field

The Lagrangian density for a massive spin field is

$$\mathcal{L} = i\bar{\psi} \partial \psi - m\bar{\psi}\psi, \quad (4.79)$$

where we recall that in the standard representation of the Dirac gamma matrices the adjoint spinor $\bar{\psi} = \psi^\dagger \gamma^0$. Rather like the complex scalar field one can obtain the field equation (which is the Dirac equation) by treating $\bar{\psi}$ and ψ as independent quantities, and demand that the action be stationary with respect to arbitrary variations in either. The variation of the action with respect to $\bar{\psi}$ gives

$$\delta S = \int d^4x \delta\bar{\psi} \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0, \quad (4.80)$$

(the Lagrangian is not a function of $\partial_\mu \bar{\psi}$ in this formulation).

Hence

$$\int d^4x \delta\bar{\psi} (i\partial\psi - m\psi) = 0, \quad (4.81)$$

From which we derive the Dirac equation $i\partial\psi - m\psi = 0$. Similarly, we can vary with respect to ψ and obtain

$$\delta S = \int d^4x (i\bar{\psi} \partial \delta\psi - m\bar{\psi} \delta\psi). \quad (4.82)$$

Integrate by parts we find

$$\delta S = \int d^4x (-i\partial_\mu \bar{\psi} \gamma^\mu - m\bar{\psi}) \delta\psi + \int dS_\mu i\bar{\psi} \gamma^\mu \delta\psi, \quad (4.83)$$

where the last term is an integral over the space-time surface at spatial infinity ($|x| \rightarrow \infty$) with end-caps at $|t| \rightarrow \infty$. As usual, we suppose that the variations die away at infinity so that we can drop the surface term, so we recover the equation for the adjoint spinor $i\partial_\mu \bar{\psi} \gamma^\mu + m\bar{\psi} = 0$.

The momentum conjugate to ψ is given by

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\bar{\psi} \gamma^0, \quad (4.84)$$

which shows that in the standard representation ψ and $i\psi^\dagger$ are canonically conjugate variables. Thus we can find the Hamiltonian density:

$$H = \pi \dot{\psi} - \mathcal{L} = -i\bar{\psi} \gamma^k \partial_k \psi + m\bar{\psi} \psi. \quad (4.85)$$

The quantization of the spinor field follows a familiar pattern. We first of all suppose that $\psi(x)$ is an operator satisfying some commutation relations, acting on some space of quantum states. We specify the equal time commutation relations, and then try to find the possible states. The problem with the spinor field is that it is not obvious from the outset what

commutation relations to impose on the field operator, and the obvious relation $[\psi(t, x), i\psi^\dagger(t, x')] = i\delta(x - x')$ is not in fact correct.

We can expand the field in terms of its operator-valued Fourier coefficients:

$$\psi(x) = \sum_{A=\pm} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E} \left(c_A(\mathbf{p}) u_A(\mathbf{p}) e^{-ip \cdot x} + d_A^\dagger(\mathbf{p}) v_A(\mathbf{p}) e^{ip \cdot x} \right), \quad (4.86)$$

where $E^2 = \mathbf{p}^2 + m^2$, $u_A(\mathbf{p})$, and $v_A(\mathbf{p})$ are 4-component spinors which satisfy

$$(p - m)u_A(\mathbf{p}) = 0, \quad (p + m)v_A(\mathbf{p}) = 0. \quad (4.87)$$

Similarly, the conjugate spinor has the expansion

$$\bar{\psi}(x) = \sum_{A=\pm} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E} \left(c_A^\dagger(\mathbf{p}) \bar{u}_A(\mathbf{p}) e^{ip \cdot x} + d_A(\mathbf{p}) \bar{v}_A(\mathbf{p}) e^{-ip \cdot x} \right). \quad (4.88)$$

We can now compute the Hamiltonian, which is the spatial integral of the Hamiltonian density H , or

$$H = \int d^3x (-i\bar{\psi}\gamma^i \partial_i \psi + m\bar{\psi}\psi). \quad (4.89)$$

The end result is

$$H = \sum_A \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E} E \left(c_A^\dagger(\mathbf{p}) c_A(\mathbf{p}) - d_A(\mathbf{p}) d_A^\dagger(\mathbf{p}) \right). \quad (4.90)$$

4.3.2 Hamilton-Jacobi Quantization of The Electromagnetic Field Coupled to A Spinor

We analyze the case of the electromagnetic field coupled to a spinor, whose Hamiltonian formalism was analyzed in Refs. [17,18]. We will consider the Lagrangian density written as

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\bar{\psi}\psi, \quad (4.91)$$

where A_μ are even variables while ψ and $\bar{\psi}$ are odd ones. The electromagnetic tensor is defined as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and we are adopting the Minkowski metric $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$. The Lagrangian function (4.91) is singular, since the rank of the Hess matrix (1.16) is three. The momenta variables conjugated, respectively, to A_i, A_0, ψ and $\bar{\psi}$, are

$$\pi^i = \frac{\overrightarrow{\partial} L}{\partial \dot{A}_i} = -F^{0i}, \quad (4.92)$$

$$\pi^0 = \frac{\overrightarrow{\partial} L}{\partial \dot{A}_0} = 0 = -H_1, \quad (4.93)$$

$$p_\psi = \frac{\overrightarrow{\partial} L}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0 = -H_\psi, \quad (4.94)$$

$$p_{\bar{\psi}} = \frac{\overrightarrow{\partial} L}{\partial \dot{\bar{\psi}}} = 0 = -H_{\bar{\psi}}. \quad (4.95)$$

With the aid of relation (4.92), the Lagrangian density may be written as

$$L = -\frac{1}{2} \pi_i \pi^i - \frac{1}{4} F_{ij} F^{ij} + i\bar{\psi} \gamma^\mu (\partial_\mu + ieA_\mu) \psi - m\bar{\psi} \psi, \quad (4.96)$$

then the canonical Hamiltonian density takes the form

$$H_0 = \pi^i \dot{A}_i + \frac{1}{2} \pi_i \pi^i + \frac{1}{4} F^{ij} F_{ij} - i\bar{\psi} (\gamma^\mu ieA_\mu + \gamma^i \partial_i) \psi + m\bar{\psi} \psi. \quad (4.97)$$

The velocities \dot{A}_i can be expressed in terms of the momenta π_i as

$$\dot{A}_i = -\pi_i + \partial_i A_0. \quad (4.98)$$

Therefore, the Hamiltonian density is

$$H_0 = \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi_i \pi^i + \pi^i \partial_i A_0 + \bar{\psi} \gamma^\mu e A_\mu \psi - \bar{\psi} (i\gamma^i \partial_i - m) \psi. \quad (4.99)$$

The set of Hamilton-Jacobi Partial Differential Equations (HJPDE) reads

$$H'_0 = p_0 + \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi_i \pi^i + \pi^i \partial_i A_0 + \bar{\psi} \gamma^\mu e A_\mu \psi - \bar{\psi} (i\gamma^i \partial_i - m) \psi, \quad (4.100)$$

$$H'_1 = \pi^0 + H_1 = \pi_0 = 0, \quad (4.101)$$

$$H'_\psi = p_\psi + H_\psi = p_\psi - i\bar{\psi} \gamma^0 = 0, \quad (4.102)$$

$$H'_{\bar{\psi}} = p_{\bar{\psi}} + H_{\bar{\psi}} = p_{\bar{\psi}} = 0. \quad (4.103)$$

Therefore, the total differential equations for the characteristic (1.41 - 1.43), are obtained as

$$\begin{aligned} dA^i &= \frac{\partial H'_0}{\partial \pi_i} dt + \frac{\partial H'_1}{\partial \pi_i} dA^0 + \frac{\partial H'_\psi}{\partial \pi_i} d\psi + \frac{\partial H'_{\bar{\psi}}}{\partial \pi_i} d\bar{\psi}, \\ &= -(\pi^i + \partial_i A_0) dt, \end{aligned} \quad (4.104)$$

$$\begin{aligned} dA^0 &= \frac{\partial H'_0}{\partial \pi_0} dt + \frac{\partial H'_1}{\partial \pi_0} dA^0 + \frac{\partial H'_\psi}{\partial \pi_0} d\psi + \frac{\partial H'_{\bar{\psi}}}{\partial \pi_0} d\bar{\psi}, \\ &= dA^0, \end{aligned} \quad (4.105)$$

$$\begin{aligned} d\pi^i &= -\frac{\partial H'_0}{\partial A_i} dt - \frac{\partial H'_1}{\partial A_i} dA^0 - \frac{\partial H'_\psi}{\partial A_i} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial A_i} d\bar{\psi}, \\ &= (\partial_i F^{li} - e\bar{\psi}\gamma^i\psi) dt, \end{aligned} \quad (4.106)$$

$$\begin{aligned} d\pi^0 &= -\frac{\partial H'_0}{\partial A_0} dt - \frac{\partial H'_1}{\partial A_0} dA^0 - \frac{\partial H'_\psi}{\partial A_0} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial A_0} d\bar{\psi}, \\ &= (\partial_i \pi^i - e\bar{\psi}\gamma^0\psi) dt, \end{aligned} \quad (4.107)$$

$$\begin{aligned} dp_\psi &= -\frac{\partial H'_0}{\partial \psi} dt - \frac{\partial H'_1}{\partial \psi} dA^0 - \frac{\partial H'_\psi}{\partial \psi} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial \psi} d\bar{\psi}, \\ &= -(i\gamma^i \partial_i + e\gamma^\mu A_\mu + m)\bar{\psi} dt, \end{aligned} \quad (4.108)$$

and

$$\begin{aligned}
dp_{\bar{\psi}} &= -\frac{\partial H'_0}{\partial \bar{\psi}} dt - \frac{\partial H'_1}{\partial \bar{\psi}} dA^0 - \frac{\partial H'_\psi}{\partial \bar{\psi}} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial \bar{\psi}} d\bar{\psi}, \\
&= (-i\gamma^i \partial_i + e\gamma^\mu A_\mu + m)\psi dt - i\gamma^0 d\psi. \tag{4.109}
\end{aligned}$$

The integrability condition ($dH'_\alpha = 0$) imply that the variations of the constraints H'_1, H'_ψ and $H'_{\bar{\psi}}$ should be identically zero

$$dH'_1 = d\pi_0 = 0, \tag{4.110}$$

$$dH'_\psi = dp_\psi - i\gamma^0 d\bar{\psi} = 0, \tag{4.111}$$

$$dH'_{\bar{\psi}} = dp_{\bar{\psi}} = 0, \tag{4.112}$$

when we substituting from Eqs. (4.108) and (4.109) into Eqs.(4.111) and (4.112) respectively, we obtain

$$dH'_\psi = 0, \tag{4.113}$$

and

$$dH'_{\bar{\psi}} = 0, \tag{4.114}$$

if and only if the relations

$$i\bar{\psi}\gamma^\mu(\overleftarrow{\partial}_\mu - ieA_\mu) + m\bar{\psi} = 0, \tag{4.115}$$

and

$$i(\partial_\mu + ieA_\mu)\gamma^\mu\psi - m\psi = 0, \tag{4.116}$$

are satisfied. Then the set of equations (4.104, 4.106, 4.107) is integrable and the equations (4.104, 4.106, 4.107) are just ordinary differential equations and can be set in the form

$$\dot{A}^i = -\pi^i - \partial_i A_0, \quad (4.117)$$

$$\dot{\pi}^i = \partial_l F^{li} - e\bar{\psi}\gamma^i\psi, \quad (4.118)$$

$$\dot{\pi}^0 = \partial_i \pi^i - e\bar{\psi}\gamma^0\psi. \quad (4.119)$$

These are the equations of motion with full gauge freedom. It can be seen, from Eq. (4.105), that A^0 is an arbitrary (gauge dependent) variable since its time derivative is arbitrary. Besides that, Eq. (4.117) shows the gauge dependence of A^i . Taking the curl of its A vector, leads to the known Maxwell equation

$$\frac{\partial \vec{A}}{\partial t} = -\vec{E} - \vec{\nabla}(A_0 - \alpha) \Rightarrow \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}. \quad (4.120)$$

Writing $j^\mu = e\bar{\psi}\gamma^\mu\psi$ we get, from Eq. (4.118), the inhomogeneous Maxwell equation

$$\frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B} - \vec{j}, \quad (4.121)$$

while the other inhomogeneous equation

$$\vec{\nabla} \cdot \vec{E} = j^0, \quad (4.122)$$

follows from Eq. (4.119). Expressions (4.115) and (4.116) are the known equations for the spinor ψ and $\bar{\psi}$.

Eqs. (1.44) and (4.100 - 4.103) lead us to the canonical action integral as

$$Z = \int d^4x \left(-\frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} \pi_i \pi^i + \pi^i \dot{A}_i + \pi^i \partial_i A_0 + i\bar{\psi} \gamma^\mu (\partial_\mu + ieA_\mu) \psi - m\bar{\psi} \psi \right). \quad (4.123)$$

Making use of equations (3.5) and (4.123), we obtain the path integral as

$$\begin{aligned} \langle out|S|In \rangle = \int \prod_i dA^i d\pi^i d\psi d\bar{\psi} \exp \left[i \left\{ \int d^4x \right. \right. \\ \left. \left(-\frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} \pi_i \pi^i + \pi^i \dot{A}_i + \pi^i \partial_i A_0 + i\bar{\psi} \gamma^\mu (\partial_\mu + ieA_\mu) \psi - m\bar{\psi} \psi \right) \right\} \right]. \end{aligned} \quad (4.124)$$

Integration over π_i gives

$$\begin{aligned} \langle out|S|In \rangle = N \int \prod_i dA^i d\psi d\bar{\psi} \exp \left[i \left\{ \int d^4x \right. \right. \\ \left. \left(\frac{1}{2} (\dot{A}^i + \partial_i A_0)^2 - \frac{1}{4} F^{ij} F_{ij} + i\bar{\psi} \gamma^\mu (\partial_\mu + ieA_\mu) \psi - m\bar{\psi} \psi \right) \right\} \right]. \end{aligned} \quad (4.125)$$

Chapter 5

Conclusion

This work is aimed at study of constrained systems in field theory using both Dirac approach and the Hamilton-Jacobi approach. The two methods, represent the Hamiltonian treatment of the constrained systems.

Dirac's approach hinges on introducing primary constraints, then constructing the total Hamiltonian by adding the primary constraints, multiplied by Lagrangian multipliers, to the usual Hamiltonian. The consistency conditions are checked on the primary constrained. All other constraints are obtained from these conditions. These constraints are classified into two types: First and second-class constraints. The distinction between these two types is quite important, not only in classical, but also in quantum theories, whenever the system having first-

class constraints, but not second-class constraints, we quantize this system by Faddeev-Popov method, but if the system having both first and second-class constraints, we quantize the system by Senjanovic method. The equations of motion, obtained using Poisson brackets, are in ordinary differential equations forms. The gauge fixing conditions, which are not an easy task in this approach, are necessary in order to determine the unknown Lagrange multipliers.

The Hamilton-Jacobi formulation (canonical method) of singular systems arrived to important result in physics, is that we first exhibit the fact that a singular system can be treated as a system with many independent variable. In other words, the equations of motion are not ordinary differential equations but total differential ones in many variables. In general mathematically speaking, it is not possible to solve the equations of motion for singular systems unless they satisfy the integrability conditions. If these conditions are not identically satisfied, it will be considered as new constraints. This process will continue until we obtain a complete system and the path integral quantization can be constructed as an integration over the canonical phase space coordinates (q_a, p_a) . The gauge fixing condition are not

necessary in the canonical formulation since one does not need to introduce Lagrange multipliers.

The integrability conditions were shown to be equivalent to the necessity of the vanishing of the variation of each H'_α , i.e. $dH'_\alpha = 0$.

The previous two methods have been applied classically in chapter two, where as we investigated three different systems; Lagrangian for a fermionic and a scalar field, the scalar field coupled to two Flavours of fermions through Yukawa couplings, and a Non-Abelian yang-Mills theories. The final results of the two approach, for every system, are found the same, and the Hamilton-Jacobi approach simpler than Dirac's approach.

Path integral quantization have been applied in chapter four, in which we investigated three different systems using the Hamilton-Jacobi quantization.

In the first system which has the scalar field coupled minimally to the vector potential, the integrability conditions dH'_0 and dH' are satisfied, the system is integrable, hence the path integral is obtained directly as an integration over the canonical phase space coordinates $A^i, \pi^i, \varphi, p_\varphi, \varphi^*$ and p_{φ^*} , without using any gauge fixing conditions.

In the second system, which has the relativistic local free field theory, the integrability conditions dH' and $d\bar{H}'$ are satisfied, so this system is integrable, and hence the path integral is obtained directly as an integration over the canonical phase-space coordinates $(\psi, \bar{\psi})$. In the usual formulation [8] one has to integrate over the extended phase-space $(p, \psi, \bar{p}, \bar{\psi})$ and one can get rid of the redundant variables (p, \bar{p}) by using delta function $\delta(p - i\frac{\lambda+1}{2}\bar{\psi}\gamma^0)$ and $\delta(\bar{p} - i\frac{\lambda-1}{2}\gamma^0\psi)$.

In the third system which has the electromagnetic field coupled to a spinor, the integrability conditions dH'_1 , dH'_ψ and $dH'_{\bar{\psi}}$ are identically satisfied, and the system is integrable. Hence, the canonical phase space coordinates (A^i, π^i) , (ψ, p_ψ) and $(\bar{\psi}, p_{\bar{\psi}})$ are obtained in terms of the parameter τ . The path integral is obtained as an integration over the canonical phase-space coordinates (A^i, π^i) and $(\psi, \bar{\psi})$ without using any gauge fixing condition. From the equations of motion for this system, we obtained the inhomogeneous Maxwell equation.

As conclusion, it is obvious that, the Hamilton-Jacobi quantization is simpler and more economical where there is no need to enlarge the initial phase-space by introducing unphysical auxiliary fields, no need to distinguish between first and second-

class constraints, no need to introduce Lagrange multipliers, and no need to use Dirac delta function in the measure as well as no need to use gauge-fixing conditions, all that is needed is a set of Hamilton-Jacobi Partial Differential Equations and the equations of motion. If the system is integrable, then one can construct the reduced canonical phase-space. In this case the path integral is obtained directly as an integration over the canonical reduced phase space coordinates.

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